Introduction to Cosmology and Relativity Physics

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Pre-relativistic Physics

- All natural phenomena take place in the arena of space and time
- A natural phenomenon consists of a sequence of events
- Event $\longrightarrow$ something that happens at some point of space and at some moment of time
- $\longrightarrow$ Description of a phenomenon involves space and time coordinates
- Our understanding of the world has evolved over time:
    - Aristotle (4th c BCE): motion occurs when a substance tries to reach its natural place; geocentric model
    - Heraklides (4th c BCE): heliocentric model of the universe, earth's rotation on its own axis
    - Aristarchus (3rd c BCE): heliocentric model; "ancient Copernicus"
    - Archimedes (3rd c BCE): center of gravity of geometric objects
    - Eratosthenes (2nd c BCE): calculated circumference of the earth
    - Hipparchus (2nd c BCE): earth's rotation + precession, calculated distance to moon
    - Ptolemy (2nd c CE): motion of planets and moon, 'planetes'; geocentric model
    - Copernicus (16th c): heliocentric model
    - Brahe/Kepler (16/17th c): accurate planetary observations; formulated laws of planetary motion
    - Galileo (17th c): Galilean telescope; basic principles of relativity
    - Newton (17th c): universal gravitation
    - Euler (18th c): description of inertial, active and passive gravitational masses
    - Laplace (18th/19th c): first attempt to modify Newtonian gravitation; potential theory; gravitational collapse; blackholes
    - Gauß/ Riemann (19th c): differential geometry of surfaces in 2 D/n-D
    - Mach (19th c): logical positivism (relativity), no absolute space, time
    - Michelson-Morley (19th c): no aether!

Special Relativity

- Classical Mechanics $\longrightarrow$ the oldest and most celebrated branch of science
  - developed on the concepts of space and time that emerged from the observations of bodies moving with speeds very small compared with the speed of light in vacuum
  - Absolute space, in its own nature, without relation to anything external, remains always similar and immovable
  - Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external and is otherwise called *duration*
- In Newtonian (classical) mechanics
  - space has three dimensions and obeys Euclidean geometry
  - unit of length is defined as the distance between two fixed points
  - other distances are measured in terms of this standard length
  - to measure time, any periodic process may be used to construct a clock
  - space and time are independent of each other $\Longrightarrow$ the space interval between two points and the time interval between two specified events do not depend on the state of motion of observers

- Two events, which are simultaneous in one frame, are also simultaneous in all other frames $\Longrightarrow$ simultaneity is an absolute concept
- Space and time are assumed to be homogeneous and isotropic
- Homogeneity means that all points in space and all moments of time are identical
- Isotropy of space means that all the directions of space are equivalent and this property allows us to orient the axes of coordinate system in any convenient direction
- Description of a natural phenomenon requires a suitable frame of reference with respect to which the space and time coordinates are to be measured
- Among all conceivable frames of reference, the most convenient ones are those in which the laws of physics appear simple
- Back in the days of Copernicus, the fundamental inertial frame was thought to be a reference frame with its origin fixed at the center of the sun and the three axes directed towards stationary stars
    - $\longrightarrow$ heliocentric model of the Universe
    - simplified description of planetary motions
- Thus all notion of Classical Mechanics was developed based on the principles of absolute space and time

Inertial and Non-inertial Frames

- An inertial frame of reference is one in which Newton's first law (the law of inertia) holds
- A frame possessing acceleration relative to an inertial frame is called non-inertial frame
- Newton's first law is not valid in non-inertial frames, since if there is an acceleration $\mathbf{a} \neq 0$, then by Newton's second law:

$$\mathbf{F} = m\mathbf{a} \ , \tag{1.1}$$

  $\mathbf{F} \neq 0$ and the law of inertia no longer holds
- Newton's laws are invariant, i.e., unchanged in any reference frame that moves with constant velocity relative to an inertial frame

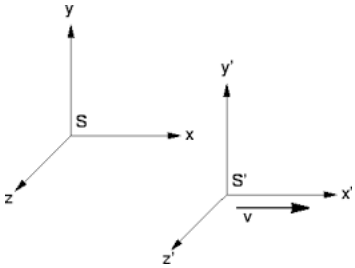


Figure 1.1: Inertial frames.

Galilean Transformations

- Between inertial frames S and S':

$$x' = x - vt\ , \quad y' = y\ , \quad z' = z\ , \quad t' = t \tag{1.2}$$

$$u'_x = u_x - v\ , \quad u'_y = u_y\ , \quad u'_z = u_z \tag{1.3}$$

$$a'_x = a_x\ , \quad a'_y = a_y\ , \quad a'_z = a_z \tag{1.4}$$

- Note the invariance

$$\mathbf{F} = m\mathbf{a} = m\mathbf{a}' = \mathbf{F}' \tag{1.5}$$

- The equation $t' = t$ is based on the assumption that time flows at the same rate in all inertial frames. This notion of time comes from our everyday experiences with slowly moving objects and is confirmed in analyzing the motion of such objects.

Einstein's Postulates

- Einstein's formulation of the theories of Special Relativity (SR) in 1905 and General Relativity (GR) around 1915 revolutionized our understanding of space and time
  - SR $\longrightarrow$ compares measurements made in inertial frames
  - GR $\longrightarrow$ accelerated frames and gravity
- As a foundation for his SR, Einstein postulated that
  1. The laws of physics are the same in all inertial reference frames
  2. The speed of light in a vacuum is equal to the value $c = 299,792,458 m/s \approx 3.00 \times 10^8 m/s$, independent of the motion of the source
- Postulate 1 generalizes Newtonian principle of relative motions to include all kinds of physical measurements
- Postulate 2 describes a common property of all electromagnetic waves

Lorentz Transformations

- For a boost in the x-direction

$$x' = \gamma\,(x - vt)\ , \qquad y' = y$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)\ , \quad z' = z \tag{1.6}$$

- Relativistic velocities transform as

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \tag{1.7}$$
$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{1.8}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{1.9}$$

- Time dilation and length contraction

$$\Delta t = \gamma\Delta t' = \gamma\tau \tag{1.10}$$
$$\Delta\ell = \frac{1}{\gamma}\Delta\ell' = \frac{1}{\gamma}\ell_p \tag{1.11}$$

Relativistic Momentum and Energy

- Relativistic correction to Newton's second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d\left(\gamma m \mathbf{u}\right)}{dt} \tag{1.12}$$

- Energy and momentum Lorentz transformations

$$E' = \gamma\left(E - v p_x\right) \tag{1.13}$$

$$p'_x = \gamma\left(p_x - vE/c^2\right) \tag{1.14}$$

$$p'_y = p_y \;, \qquad p'_z = p_z \tag{1.15}$$

The Spacetime Interval and Invariant Mass

- Space and time are linearly dependent on each other in the Lorentz transformation
- Spacetime $\longrightarrow$ 4-dimensional relativistic world
- The spacetime interval between two events

$$(\Delta s)^2 \equiv \Delta s^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] \tag{1.16}$$

- The spacetime interval Δs is the 4-D analog of space interval $\mathbf{r}$ in 3-D
- $\Delta\mathbf{s}$ is a 4-vector with components $c\Delta t$, Δx , Δy , Δz
- [Ex] Δs is invariant under a Lorentz transformation in spacetime (show!)
- The negative sign in Eq. (1.16) implies that Δs^2 may be positive, negative, or zero depending on the relative sizes of the time and space separations

- Any quantity that transforms like $\Delta \mathbf{s}$, i.e., is invariant under Lorentz transformation, is also a 4-vector
- Time-like intervals:

$$(\Delta s)^2 = c^2 \tau^2 \implies \tau = \Delta s / c \longrightarrow \text{proper time interval} \tag{1.17}$$

- Space-like intervals:

$$\begin{aligned} (\Delta s)^2 &= \left[\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] \\ \implies L_p = \Delta s &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \longrightarrow \text{proper length} \end{aligned} \tag{1.18}$$

- Light-like intervals:

$$(\Delta s)^2 = 0 \tag{1.19}$$

- In the limiting cases $\Delta t \to 0$, $\Delta \mathbf{r} \to 0$, we write

$$ds^2 = c^2 dt^2 - \left[dx^2 + dy^2 + dz^2\right] \tag{1.20}$$

- Note the spacetime signature: $(+ - --)$ in this case

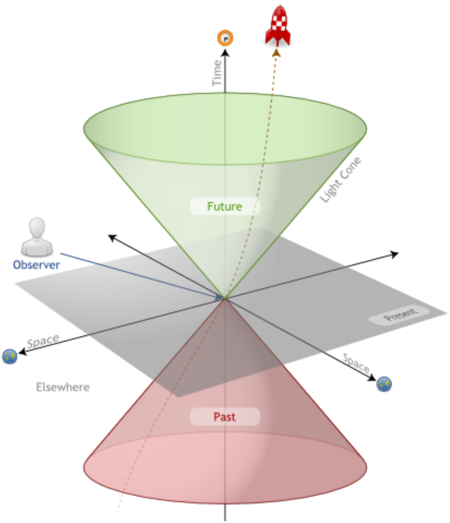


Figure 1.2: The spacetime diagram.

- With the sign of Δs^2, nature is telling us about the causal relation between events

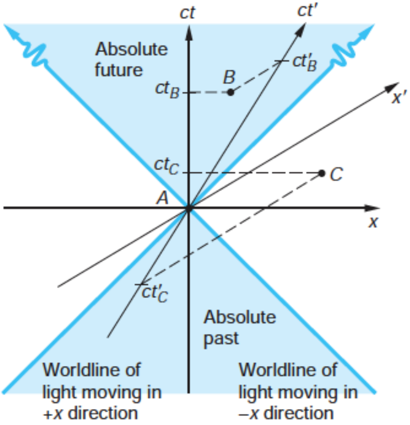


Figure 1.3: Time-like, space-like and light-like spacetime regions. Only time-like regions preserve causality!

- Notice that whichever of the three possibilities characterizes a pair for one observer, it does so for all observers since Δs is invariant
- The relative temporal order of events for pairs characterized by timelike intervals, such as A and B, is the same for all inertial observers
- Events in the upper shaded area will all occur in the future of A; those in the lower shaded area occurred in A's past
- Events whose intervals are spacelike, such as A and C, can be measured as occurring in either order, depending on the relative motion of the frames. Thus, C occurs after A in S, but before A in S'
- [Ex] Using the definitions of relativistic momentum and relativistic energy, we can write the rest energy of the mass m as the invariant energy-momentum 4-vector (show!):

$$(mc^2)^2 = E^2 - (pc)^2 \tag{1.21}$$

- $\longrightarrow$ Observers in all inertial frames measure the same value for the rest energy of isolated systems and, and therefore the same value for the mass

General Relativity

- The equivalence principle(s)

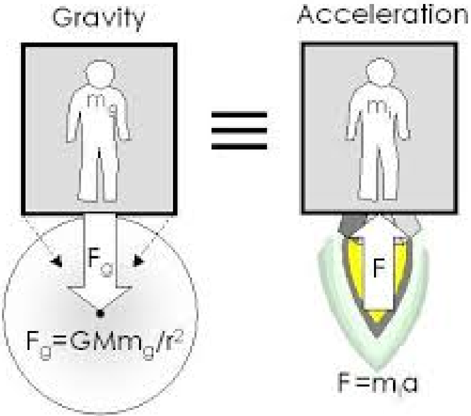


Figure 1.4: The gravitational field cannot be distinguished from a suitably chosen accelerated reference frame.

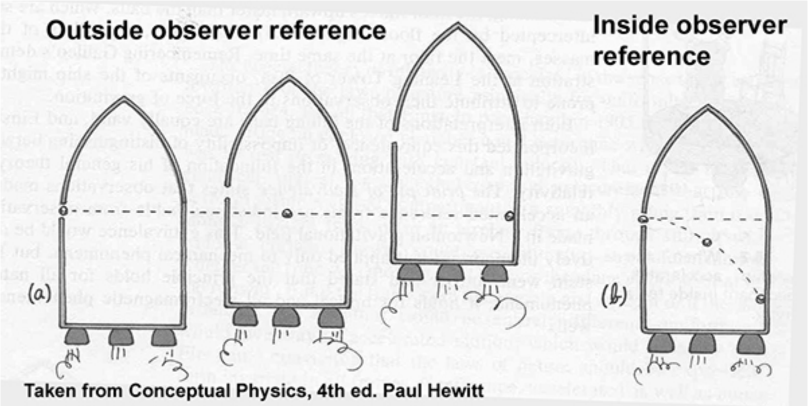


Figure 1.5: The bending of light in an accelerated reference frame.

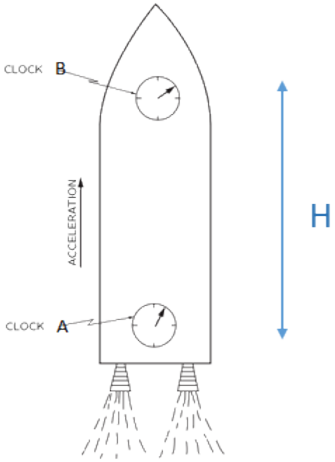


Figure 1.6: Gravitational redshift. Clocks tick slower in a stronger gravitational field.

- The generalization of relativity to accelerated, *i.e.*, noninertial reference frames by Einstein in 1915 is known as the General Theory of Relativity or, commonly, General Relativity (GR)
- It is the theory that describes gravity, one of the four fundamental forces of nature

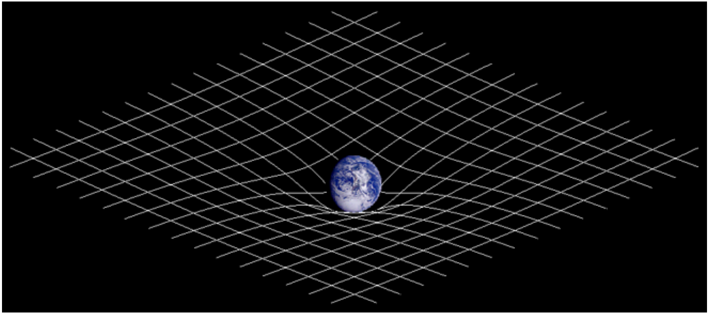


Figure 1.7: Matter tells spacetime how to curve, spacetime tells matter how to move.

- GR is the basis of our understanding of the Big Bang...

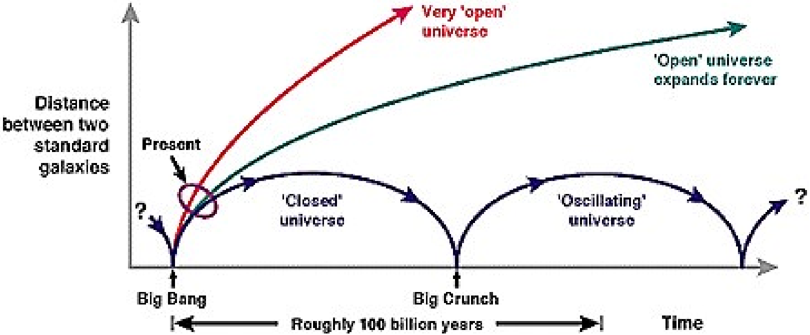


Figure 1.8: Big Bang and cosmic evolution.

- … black holes

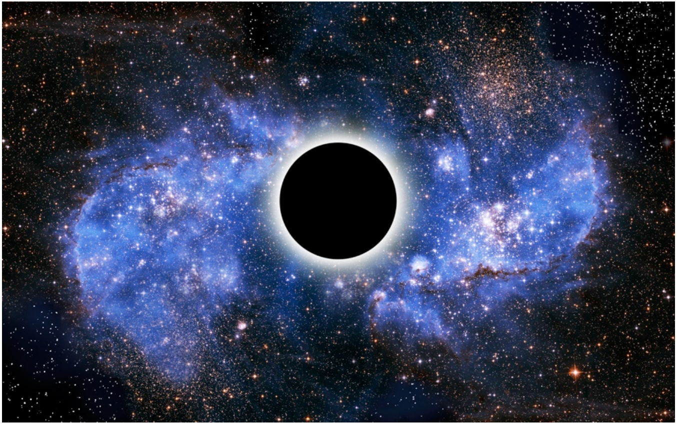


Figure 1.9: Superdense regions in spacetime that even light cannot escape.

- … quasars

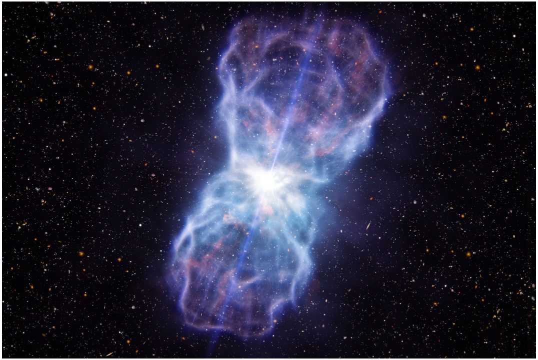


Figure 1.10: Farthest, brightest objects ever seen. Appear as stars in telescopes, hence the term *quasar*=quasi-stellar.

- ...the life cycle of stars

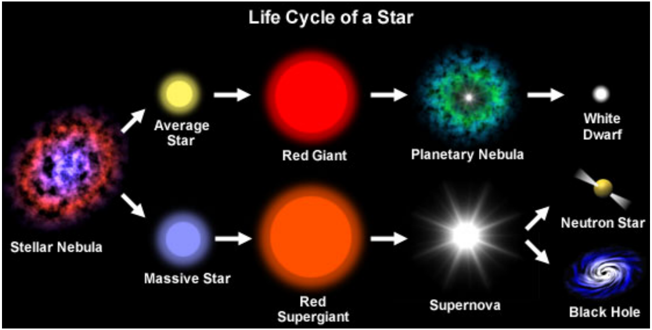


Figure 1.11: Stellar Evolution.

- … the evolution of the Universe as a whole

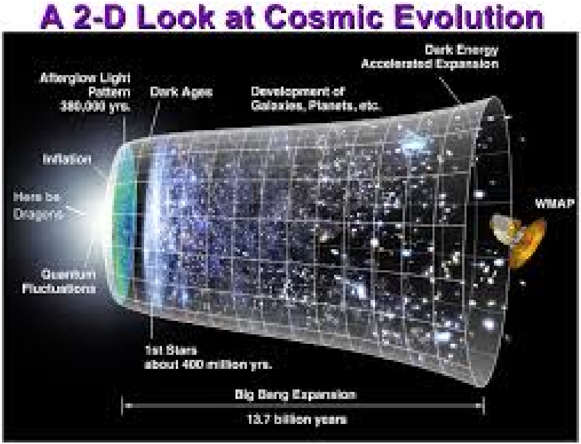


Figure 1.12: The evolution of the Universe.

Tensors

- Any physical quantity is determined by a se of numerical values, its components, which depend on the coordinate system
- Scalars $\rightarrow$ determined by one numerical value, independent of coordinate system
- Vectors $\rightarrow$ covariant and contravariant types

Example: displacement in Euclidean space

- Suppose the Cartesian coordinates of points A and B are given by x_A^α and x_B^α, $\alpha = 1, 2, 3$ (x,y,z). Consider the displacement **AB**. The components of this displacement vector are given by

$$AB = x_B^\alpha - x_A^\alpha \tag{1.22}$$

  - Definition not valid in non-Cartesian coordinates
  - Only infinitesimal displacements can be defined this way: from point A with coordinates x^α to a neighboring point A' with coordinates $x^\alpha + dx^\alpha$
  - Components of vector=difference of coordinates of A and A' :

$$AA' = dx^\alpha \tag{1.23}$$

- Def. (1.23) is valid for general coordinates in any n-dimensional space
- For this course $n = 4$ and [1]

$$t \equiv x^0 \ , \ x \equiv x^1 \ , \ y \equiv x^2 \ , \ z \equiv x^3 \ . \tag{1.24}$$

[1] Unless otherwise stated, Latin indices i, j etc will denote spatial coordinates and run from 1 to 3; Greek indices α, β etc run from 0 to 3 and denote spacetime coordinates.

- Consider a smooth curve passing through the points A and A' , determined by n functions of a scalar parameter

$$x^\alpha = f^\alpha(\lambda) \tag{1.25}$$

- If A and A' correspond to the values λ and $\lambda + d\lambda$ of the parameter, then the tangent vector v^α to the curve at A has the components

$$v^\alpha = \frac{dx^\alpha}{d\lambda} \tag{1.26}$$

- dx^α and $v^\alpha \to$ prototypes of contravariant vectors
- Coordinate transformations from initial x^α to new $\tilde{x}^\alpha$ is determined by n equations of the form

$$\tilde{x}^\alpha = f^\alpha(x^\beta) \tag{1.27}$$

- Then

$$d\tilde{x}^\alpha = \sum_\beta \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} dx^\beta \tag{1.28}$$

- Components of $d\tilde{x}^\alpha$ in the coordinate system $\tilde{x}^\alpha$ of the infinitesimal displacement $\mathbf{AA}'$ whch has the components dx^α in x^α
- λ scalar $\implies$ $d\lambda$ has the same value in both coordinate systems

$$\tilde{v}^\alpha = \sum_\beta \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} v^\beta \tag{1.29}$$

- Eq. (1.29) is the general transformation law for contravariant vectors

- Note that

$$\sum_{\beta} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\gamma}} = \delta^{\alpha}_{\gamma} = \sum_{\beta} \frac{\partial x^{\alpha}}{\partial x^{\tilde{\beta}}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\gamma}} \tag{1.30}$$

where the Kronecker (delta) symbol

$$\delta^{\alpha}_{\gamma} = \begin{cases} 1 & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise} \end{cases}$$

- One can show that

$$dx^{\alpha} = \sum_{\alpha} \frac{\partial x^{\beta}}{\partial \tilde{x}^{\alpha}} d\tilde{x}^{\alpha} \tag{1.31}$$

provided

$$det\left(\frac{\partial \tilde{x}^{\alpha}}{\partial x^{\beta}}\right) \neq 0, \infty \tag{1.32}$$

- A covariant vector b_{α} is a quantity whose n components depend on the coordinate system in such a way that, if a^{α} is any contravariant vector, then the sum

$$\sum_{\alpha} b_{\alpha} a^{\alpha} \rightarrow \text{scalar} \tag{1.33}$$

$$\sum_{\alpha} b_{\alpha} a^{\alpha} = \sum_{\alpha} \tilde{b}_{\alpha} \tilde{a}^{\alpha} \text{ for any } x^{\alpha} \rightarrow \tilde{x}^{\alpha} \quad \text{(scalar product)} \tag{1.34}$$

- Note that from (1.29) and (1.34) follows

$$\sum_{\beta} a^{\beta}\left[b_{\beta}-\sum_{\alpha}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\beta}}\tilde{b}_{\alpha}\right]=0 \implies b_{\beta}=\sum_{\alpha}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\beta}}\tilde{b}_{\alpha} \tag{1.35}$$

- Using (1.30) and (1.35), we can solve for $\tilde{b}_{\alpha}$:

$$\tilde{b}_{\alpha}=\sum_{\beta}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\alpha}}b_{\beta} \tag{1.36}$$

- Tensor equations: the same form in any coordinate system
- Rank /order:The total number of contravariant and covariant indices of a tensor
  - Independent of the number of dimensions N of the underlying space
  - Scalar: 1 ($=3^0$) component (magnitude only) $\rightarrow$ *Tensor of rank* 0
  - Vector: 3 ($=3^1$) components (one direction and magnitude) $\rightarrow$ *Tensor of rank* 1
  - Dyad, N$\times$N matrix: 9 ($=3^2$) components (two directions and magnitude) $\rightarrow$ *Tensor of rank* 2
  - Triad: 27 ($=3^3$) components (three directions and magnitude) $\rightarrow$ *Tensor of rank* 3
  - Tetrad: 81 ($=3^4$) components (four directions and magnitude) $\rightarrow$ *Tensor of rank* 4

Einstein Summation

- Convention: sum over repeated indices

$$T^{\cdots}_{\cdots\alpha} V^{\cdots\alpha}_{\cdots} \equiv \sum_{\alpha} T^{\cdots}_{\cdots\alpha} V^{\cdots\alpha}_{\cdots} \tag{1.37}$$

- Thus

$$\tilde{b}_{\alpha} = \frac{\partial x^{\beta}}{\partial \tilde{x}^{\alpha}} b_{\beta} = \Lambda_{\alpha}{}^{\beta} b_{\beta} \tag{1.38}$$

$$\tilde{a}^{\alpha} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\beta}} a^{\beta} = \Lambda^{\alpha}{}_{\beta} a^{\beta} \tag{1.39}$$

Contraventariant and Covariant Tensors

- A contravariant tensor of order 2
  - has n^2 components $T^{\gamma\alpha}$
  - transforms, when $x^\alpha \to \tilde{x}^\alpha$, in such a way that, for arbitrary covariant vectors a_α , b_α, the sum

$$T^{\gamma\alpha} a_\gamma b_\alpha \to \quad \text{scalar} \tag{1.40}$$

$$T^{\gamma\alpha} a_\gamma b_\alpha = \tilde{T}^{\gamma\alpha} \tilde{a}_\gamma \tilde{b}_\alpha \tag{1.41}$$

- Similarly, a covariant tensor of order 2 is defined by the condition (for any $x^\alpha \to \tilde{x}^\alpha$ and arbitrary vectors a^γ and b^α)

$$T_{\gamma\alpha} a^\gamma b^\alpha = \tilde{T}_{\gamma\alpha} \tilde{a}^\gamma \tilde{b}^\alpha \tag{1.42}$$

- A mixed tensor of order 2

$$T^\gamma_\alpha a_\gamma b^\alpha = \tilde{T}^\gamma_\alpha \tilde{a}_\gamma \tilde{b}^\alpha \tag{1.43}$$

- Transformation rules for tensors:

$$\tilde{T}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial \tilde{x}^\beta}{\partial x^\delta} T^{\gamma\delta} = \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta T^{\gamma\delta} \tag{1.44}$$

$$\tilde{T}_{\alpha\beta} = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} T_{\gamma\delta} = \Lambda_\alpha{}^\gamma \Lambda_\beta{}^\delta T_{\gamma\delta} \tag{1.45}$$

$$\tilde{T}^\alpha_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial \tilde{x}^\beta} T^\gamma_\delta = \Lambda^\alpha{}_\gamma \Lambda_\beta{}^\delta T^\gamma_\delta \tag{1.46}$$

[Ex] Show that the Kronecker symbol is a mixed tensor.

General Tensors

- Tensor of order (m, n) , $m, n \geq 0$:

$$T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} \tag{1.47}$$

Scalar $\rightarrow$ tensor of order (0,0)
Contravariant vector $\rightarrow$ tensor of order (1,0)
Covariant vector $\rightarrow$ tensor of order (0,1)
Kronecker symbol $\rightarrow$ tensor of order (1,1)

- If a_{α_1} , a_{α_2} , ...a_{α_m} and b^{β_1} , b^{β_2} , ...b^{β_n} are m and n arbitrary covariant and contravariant vectors, then the sum

$$T^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_m} b^{\beta_1} b^{\beta_2} \ldots b^{\beta_n} \rightarrow \text{scalar} \tag{1.48}$$

Tensor Operations

- Only tensors of the same order can be added
- The product of 2 vectors is a tensor of order 2.
  [Ex] Show that $a^\alpha b^\beta$ is a tensor of order $(2,0)$.
- Two tensors of any order can be multiplied
- Product of tensors of order (m,n) and (m',n') is a tensor of order $(m+m',n+n')$
- A contraction of any tensor of order (m,n) , $m,n>0$ is allowed and results in a tensor of order $(m-1,n-1)$.
  [Ex] Using the transformation

$$\tilde{T}^{\alpha\beta}_{\gamma} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\frac{\partial x^\sigma}{\partial \tilde{x}^\gamma} T^{\mu\nu}_{\sigma} \ , \tag{1.49}$$

  show that setting $\gamma=\beta$ in $T^{\alpha\beta}_{\gamma}$ results in a contravariant vector.
  [Ex] Show that T^α_α is a scalar.[2]

[2] This scalar is the trace of the mixed tensor T^α_β.

Tensor Operations...

- Matrix representation of the components of a contravariant tensor:

$$T^{\alpha\beta} = \begin{pmatrix} T^{11} & T^{12} & \ldots & T^{1n} \\ T^{21} & T^{22} & \ldots & T^{2n} \\ \ldots & \ldots & \ldots & \ldots \\ T^{n1} & T^{n2} & \ldots & T^{nn} \end{pmatrix}$$

- Symmetric tensor:

$$T^{\alpha\beta} = T^{\beta\alpha} \,, \qquad T_{\alpha\beta} = T_{\beta\alpha} \tag{1.50}$$

- Antisymmetric tensor:

$$T^{\alpha\beta} = -T^{\beta\alpha} \qquad T_{\alpha\beta} = -T_{\beta\alpha} \tag{1.51}$$

- No symmetry properties for a mixed tensor: the matrix components of T^{α}_{β} could be symmetric in some frame x^{α} but this property would not be conserved in a coordinate transformation
- A tensor of order 2 can be written as the sum of a symmetric and antisymmetric tensor:

$$T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta} \,, \quad S_{\alpha\beta} = S_{\beta\alpha} \,, \quad A_{\alpha\beta} = -A_{\beta\alpha} \tag{1.52}$$

- $S_{\alpha\beta}$ and $A_{\alpha\beta}$ are determined uniquely:

$$S_{\alpha\beta} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right) \equiv T_{(\alpha\beta)} \,, \qquad A_{\alpha\beta} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right) \equiv T_{[\alpha\beta]} \tag{1.53}$$

Tensor Operations...

- # of independent components of $T_{(\alpha\beta)}$ and $T_{[\alpha\beta]}$, respectively:

$$n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}\,, \qquad \frac{n^2 - n}{2} = \frac{n(n-1)}{2} \tag{1.54}$$

- For a tensor of rank (m, n), [anti-]symmetry is defined w.r.t a pair of indices which are both upper or lower. E.g., $T^{\alpha\beta\gamma}$ is sym./antisym. in α , β if

$$T^{\alpha\beta\gamma} = T^{\beta\alpha\gamma}\,, \qquad T^{\alpha\beta\gamma} = -T^{\beta\alpha\gamma} \tag{1.55}$$

- A tensor having only upper/lower indices is called [totally] symmetric/antisymmetric if it is sym./antisym. w.r.t. any pair of indices. E.g., $T^{\alpha\beta\gamma}$ is symmetric/antisymmetric if

$$T^{\alpha\beta\gamma} = T^{\beta\alpha\gamma} = T^{\alpha\gamma\beta} = T^{\gamma\beta\alpha}\,, \qquad T^{\alpha\beta\gamma} = -T^{\beta\alpha\gamma} = -T^{\alpha\gamma\beta} = -T^{\gamma\beta\alpha} \tag{1.56}$$

- Note

$$T_{(\alpha\beta\gamma)} = \frac{1}{6}\left(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} + T_{\gamma\beta\alpha} + T_{\beta\gamma\alpha} + T_{\alpha\gamma\beta}\right) \tag{1.57}$$

$$T_{[\alpha\beta\gamma]} = \frac{1}{6}\left(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} - T_{\gamma\beta\alpha} - T_{\beta\gamma\alpha} - T_{\alpha\gamma\beta}\right) \tag{1.58}$$

Riemannian Space

- Curved 2-D space can be thought of as a surface immersed in Euclidean 3-D space
- Curved 4-D space $\rightarrow$ immersed in a flat space of larger number of dimensions $\equiv$ Riemann(ian) space
- A small region in a Riemannian space is approximately flat
- Invariant space ds between a point x^α and a neighboring point $x^\alpha + dx^\alpha$:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1.59}$$

- If spacetime signature is $(- + ++)$

$$ds^2 \begin{cases} < 0 & \text{timelike, } ds \rightarrow \text{imaginary} \\ = 0 & \text{lightlike (null)} \\ > 0 & \text{spacelike, } ds \rightarrow \text{real} \end{cases}$$

Affine Connection

- Consider basis vectors $\mathbf{e}_\alpha$ at A and A' in x^α:

$$\mathbf{e}_\alpha(A') = \mathbf{e}_\alpha(A) + \delta\mathbf{e}_\alpha \tag{1.60}$$

- Then

$$\mathbf{e}_{\alpha,\beta} = \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \equiv \lim_{\delta x^\beta \to 0} \frac{\delta \mathbf{e}_\alpha}{\delta x^\beta} = \Gamma^\theta_{\alpha\beta}\mathbf{e}_\theta \tag{1.61}$$

- $\Gamma^\theta_{\alpha\beta} \to$ affine connection /Christoffel symbol (of the second kind) at A. One can also write

$$\Gamma^\theta_{\alpha\beta} = \mathbf{e}^\theta \mathbf{e}_{\alpha,\beta} = \frac{1}{2} g^{\theta\delta}\left[g_{\delta\beta,\alpha} + g_{\alpha\delta,\beta} - g_{\alpha\beta,\delta}\right] \tag{1.62}$$

- Sym. in α, β
- Transforms as

$$\Gamma^{\theta'}_{\alpha'\beta'} = x^{\theta'}_{,\delta}\, e^\delta \left(x^\mu_{,\alpha'} e_\mu\right)_{,\beta'} = x^{\theta'}_{,\delta}\, e^\delta \left(x^\mu_{,\alpha'} e_{\mu,\beta'} + x^\mu_{,\alpha',\beta'} e_\mu\right) \tag{1.63}$$

$$x^{\theta'}_{,\delta} x^\mu_{,\alpha'} x^\nu_{,\beta'} \Gamma^\delta_{\mu\nu} + x^{\theta'}_{,\delta} x^\delta_{,\alpha'\beta'} \implies \text{nontensor} \tag{1.64}$$

- Christoffel symbol of the first kind

$$\Gamma_{\delta\alpha\beta} = g_{\delta\theta}\Gamma^\theta_{\alpha\beta} \tag{1.65}$$

Covariant Differentiation

- For any scalar field quantity S, we know

$$S_{,\alpha'} = S_{,\beta} x^{\beta}_{,\alpha'} \tag{1.66}$$

Thus the derivative $S_{,\alpha}$ of a scalar field is a covariant vector field.

- Given a vector field V_α, is $V_{\alpha,\beta}$ also a tensor? From (1.38)

$$V_{\alpha'} = V_\gamma x^{\gamma}_{,\alpha'} \implies V_{\alpha',\beta'} = \left(V_\gamma x^{\gamma}_{,\alpha'}\right)_{,\beta'}$$

$$= V_{\gamma,\delta} x^{\delta}_{,\beta'} x^{\gamma}_{,\alpha'} + V_\gamma x^{\gamma}_{,\alpha'\beta'} \rightarrow \text{nontensor} \tag{1.67}$$

- Define the quantity

$$V_{\alpha,\beta} - \Gamma^{\delta}_{\alpha\beta} V_\delta \rightarrow \text{ transforms as a vector} \tag{1.68}$$

- Covariant derivative of V_α, and is usually written

$$V_{\alpha;\beta} \equiv V_{\alpha,\beta} - \Gamma^{\delta}_{\alpha\beta} V_\delta \tag{1.69}$$

[Ex] Show that the covariant derivative of a contravariant vector is given by

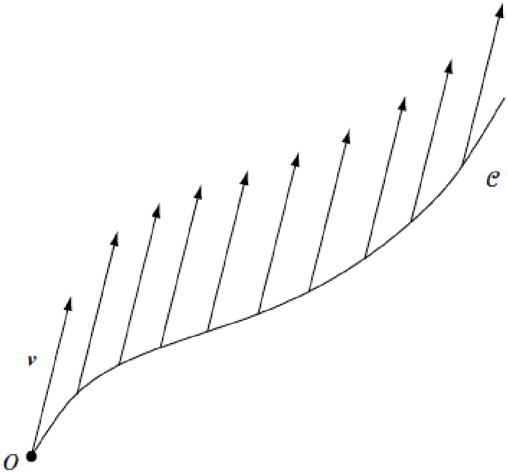
$$V^{\alpha}_{;\beta} \equiv V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\delta\beta} V^{\delta} \tag{1.70}$$

[Ex] Show that

$$T_{\alpha\beta;\delta} = T_{\alpha\beta,\delta} - \Gamma^{\theta}_{\alpha\delta} T_{\theta\beta} - \Gamma^{\sigma}_{\beta\delta} T_{\alpha\sigma} \tag{1.71}$$

Parallel Transport/ Displacement

- Consider a curve $\mathcal{C}$ parametrized by $x^\alpha(\lambda)$, with a vector field V^α defined at some initial point O



- Transport V^α along $\mathcal{C}$ with tangent

$$U^\alpha = \frac{dx^\alpha}{d\lambda} \tag{1.72}$$

such that

$$\frac{dV^\alpha}{d\lambda} = 0 \tag{1.73}$$

is satisfied at each point along $\mathcal{C}$

- Parallel field of vectors at each point along $\mathcal{C}$
    - Euclidean space $\rightarrow$ no change of length/ direction
- The condition for parallel transport in an inertial frame:

$$\frac{dV^\alpha}{d\lambda} = V^\alpha_{,\beta}\frac{dx^\beta}{d\lambda} = U^\beta V^\alpha_{,\beta} \tag{1.74}$$

- In a general frame

$$\frac{dV^\alpha}{d\lambda} = V^\alpha_{;\beta}\frac{dx^\beta}{d\lambda} = U^\beta V^\alpha_{;\beta} \tag{1.75}$$

Geodesics

- A curve that parallel transports its own tangent vector is called a geodesic:

$$U^{\beta} U^{\alpha}_{;\beta} = 0 \tag{1.76}$$

- $\rightarrow$ analogous to a 'straight line' in curved space
  [Ex] Using (1.76), show that one can derive the geodesic equation (GE):

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\beta\delta} \frac{dx^{\beta}}{d\lambda} \frac{dx^{\delta}}{d\lambda} = 0 \tag{1.77}$$

- SODE in $x^{\alpha}(\lambda)$ $\rightarrow$ unique solution requires specifying $\mathbf{x_0}$ and $\mathbf{U_0}$
  - If U^{α} is initially a null vector, i.e., if $U^{\alpha} U_{\alpha} = 0$, it will always remain a null vector $\rightarrow$ null geodesic. The path of a ray of light, e.g., is a null geodesic
  - If U^{α} is initially a timelike vector, i.e., if $U^{\alpha} U_{\alpha} < 0$, it will always remain a timelike vector $\rightarrow$ timelike geodesic
  - If U^{α} is initially a spacelike vector, i.e., if $U^{\alpha} U_{\alpha} > 0$, it will always remain a spacelike vector $\rightarrow$ spacelike geodesic

The Curvature Tensor

- Q: Does order (of two successtive operations) matter in covariant differentiation?
- For a scalar:

$$S_{;\alpha;\beta} = S_{;\alpha,\beta} - \Gamma^{\delta}_{\alpha\beta} S_{;\delta} = S_{,\alpha\beta} - \Gamma^{\delta}_{\alpha\beta} S_{,\delta} \tag{1.78}$$

- For a vector:

$$\begin{aligned} V_{\alpha;\beta;\delta} &= V_{\alpha;\beta,\delta} - \Gamma^{\theta}_{\alpha\delta} V_{\theta;\beta} - \Gamma^{\theta}_{\beta\delta} V_{\alpha;\theta} \\ &= \left(V_{\alpha,\beta} - \Gamma^{\theta}_{\alpha\beta} V_{\theta}\right)_{,\delta} - \Gamma^{\theta}_{\alpha\delta}\left(V_{\theta,\beta} - \Gamma^{\eta}_{\theta\beta} V_{\eta}\right) - \Gamma^{\theta}_{\beta\delta}\left(V_{\alpha,\theta} - \Gamma^{\eta}_{\alpha\theta} V_{\eta}\right) \\ &= V_{\alpha,\beta,\delta} - \Gamma^{\theta}_{\alpha\beta} V_{\theta,\delta} - \Gamma^{\theta}_{\alpha\delta} V_{\theta,\beta} - \Gamma^{\theta}_{\beta\delta} V_{\alpha,\theta} - V_{\eta}\left(\Gamma^{\eta}_{\alpha\beta,\delta} - \Gamma^{\theta}_{\alpha\delta}\Gamma^{\eta}_{\theta\beta} - \Gamma^{\theta}_{\beta\delta}\Gamma^{\eta}_{\alpha\theta}\right) \end{aligned} \tag{1.79}$$

Swap β and δ:

$$V_{\alpha;\delta;\beta} = V_{\alpha,\delta,\beta} - \Gamma^{\theta}_{\alpha\delta} V_{\theta,\beta} - \Gamma^{\theta}_{\alpha\beta} V_{\theta,\delta} - \Gamma^{\theta}_{\delta\beta} V_{\alpha,\theta} - V_{\eta}\left(\Gamma^{\eta}_{\alpha\delta,\beta} - \Gamma^{\theta}_{\alpha\beta}\Gamma^{\eta}_{\theta\delta} - \Gamma^{\theta}_{\delta\beta}\Gamma^{\eta}_{\alpha\theta}\right) \tag{1.80}$$

(1.79)- (1.80) gives

$$V_{\alpha;\beta;\delta} - V_{\alpha;\delta;\beta} = V_{\eta}\left(\Gamma^{\eta}_{\alpha\delta,\beta} - \Gamma^{\eta}_{\alpha\beta,\delta} + \Gamma^{\theta}_{\alpha\delta}\Gamma^{\eta}_{\theta\beta} - \Gamma^{\theta}_{\alpha\beta}\Gamma^{\eta}_{\theta\delta}\right) \tag{1.81}$$

- Def. the Riemann[-Christoffel]/ curvature tensor:

$$R^{\eta}_{\alpha\beta\delta} \equiv \Gamma^{\eta}_{\alpha\delta,\beta} - \Gamma^{\eta}_{\alpha\beta,\delta} + \Gamma^{\theta}_{\alpha\delta}\Gamma^{\eta}_{\theta\beta} - \Gamma^{\theta}_{\alpha\beta}\Gamma^{\eta}_{\theta\delta} \tag{1.82}$$

- It can be shown that

$$R^{\eta}_{\ \alpha\beta\delta} = -R^{\eta}_{\ \alpha\delta\beta} \tag{1.83}$$

$$R^{\eta}_{\ \alpha\beta\delta} + R^{\eta}_{\ \beta\delta\alpha} + R^{\eta}_{\ \delta\alpha\beta} = 0 = R^{\eta}_{\ [\alpha\beta\delta]} \tag{1.84}$$

- Since

$$R_{\theta\alpha\beta\delta} = g_{\theta\eta} R^{\eta}_{\ \alpha\beta\delta} = \frac{1}{2}\left[g_{\theta\delta,\alpha\beta} - g_{\alpha\delta,\theta\beta} - g_{\theta\beta,\alpha\delta} + g_{\alpha\beta,\theta\delta}\right] + \Gamma_{\eta\theta\delta}\Gamma^{\eta}_{\ \alpha\beta} - \Gamma_{\eta\theta\beta}\Gamma^{\eta}_{\ \alpha\delta} \tag{1.85}$$

one can also show that

$$R_{\alpha\beta\delta\theta} = -R_{\beta\alpha\delta\theta} \tag{1.86}$$

$$R_{\alpha\beta\delta\theta} = R_{\delta\theta\alpha\beta} = R_{\theta\delta\beta\alpha} \tag{1.87}$$

- $\Longrightarrow$ Only 20 out of 256 components of $R_{\alpha\beta\delta\theta}$ are independent!
  [Ex] Show:

$$R_{\alpha\beta\delta\theta} = 0 \iff \text{flat space} \tag{1.88}$$

The Bianchi Identities

$$R^{\alpha}_{\beta\delta\theta;\mu} + R^{\alpha}_{\beta\theta\mu;\delta} + R^{\alpha}_{\beta\mu\delta;\theta} = 0 = R^{\alpha}_{\beta[\delta\theta;\mu]} \tag{1.89}$$

The Ricci Tensor

$$R^{\delta}_{\alpha\delta\beta} = R_{\alpha\beta} = R_{\beta\alpha} = 2\left[\Gamma^{\delta}_{\alpha[\beta,\delta]} + \Gamma^{\delta}_{\theta[\delta}\Gamma^{\theta}_{\beta]\alpha}\right] \tag{1.90}$$

The Ricci/Curvature Scalar

$$g^{\alpha\beta}R_{\alpha\beta} = R^{\alpha}_{\alpha} = R \tag{1.91}$$

- Total curvature of spacetime
- Defined such that $R > 0$ for the surface of a sphere in 3-D

Einstein's Field Equations

- From (1.89), contracting wrt α, μ and multiplying throughout by $g^{\beta\delta}$, we get

$$g^{\beta\delta}\left[R^{\alpha}_{\beta\delta\theta;\alpha} + R^{\alpha}_{\beta\theta\alpha;\delta} + R^{\alpha}_{\beta\alpha\delta;\theta}\right] = 0 \tag{2.1}$$

- But this can be rewritten as

$$\left(g^{\beta\delta}R^{\alpha}_{\beta\delta\theta}\right)_{;\alpha} + \left(g^{\beta\delta}R^{\alpha}_{\beta\theta\alpha}\right)_{;\delta} + \left(g^{\beta\delta}R^{\alpha}_{\beta\alpha\delta}\right)_{;\theta} = 0 \tag{2.2}$$

- But since

$$g^{\beta\delta}R^{\alpha}_{\beta\delta\theta} = g^{\beta\delta}g^{\alpha\eta}R_{\eta\beta\delta\theta} = g^{\beta\delta}g^{\alpha\eta}R_{\beta\eta\theta\delta} = -g^{\alpha\eta}R_{\eta\theta} = -R^{\alpha}_{\theta} \tag{2.3}$$

(2.2) becomes

$$-R^{\alpha}_{\theta;\alpha} - \left(g^{\beta\delta}R_{\beta\theta}\right)_{;\delta} + R_{;\theta} = 0 \implies 2R^{\alpha}_{\theta;\alpha} - R_{;\theta} = 0 \tag{2.4}$$

$$\implies 2\left(R^{\alpha}_{\theta} - \frac{1}{2}g^{\alpha}_{\theta}R\right)_{;\alpha} = 0 \tag{2.5}$$

$$\implies \left(R^{\alpha\theta} - \frac{1}{2}g^{\alpha\theta}R\right)_{;\alpha} = 0 \tag{2.6}$$

- Define the Einstein tensor:

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R \tag{2.7}$$

- Einstein: gravity is a manifestation of spacetime curvature induced by the presence of matter. This is the basic assumption that lead to the development of Einstein's GR
- $\rightarrow$ Curvature of spacetime at any event is related to the matter distribution at that event
- In vacuum/ empty space (no matter, no physical fields but gravity)

$$R_{\alpha\beta} = 0 \tag{2.8}$$

- In the presence of matter with energy-momentum tensor (EMT) $T^{\alpha\beta}$, the component $T^{\alpha\beta}$ represents the flux or flow of the component of the 4-momentum crossing the surface of constant x^{β}
  - $T^{00} \rightarrow$ energy density
  - $T^{0i} \rightarrow$ flow/flux of energy in the x^i direction
  - $T^{i0} \rightarrow$ density of the i^{th} component of momentum
  - $T^{ij} \rightarrow$ the flow of the i^{th} component of momentum in the j^{th} direction
- For a perfect fluid

$$T^{\alpha\beta} = (\mu + p)u^{\alpha}u^{\beta} + pg^{\alpha\beta} \tag{2.9}$$

- For a comoving observer

$$u^{\alpha} = \delta_0^{\alpha} \tag{2.10}$$

- Thus

$$T^{\alpha\beta} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- Conservation equations for the EMT

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{2.11}$$

- We also know that the [covariant] divergence of the Einstein tensor vanishes:

$$G^{\alpha\beta}{}_{;\beta} = 0 \tag{2.12}$$

- Einstein inferred that the two tensors should be proportional and wrote

$$G^{\alpha\beta} = \kappa T^{\alpha\beta} \tag{2.13}$$

  κ being a coupling constant to be determined through the assumption that the field equations should reduce to the Poisson equation

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \mu(\mathbf{x}) \tag{2.14}$$

  in the Newtonian (nonrelativistic) limit. This assumption gives[3] (see derivation shortly)

$$\kappa = \frac{8\pi G}{c^4} \tag{2.15}$$

- The EFEs (2.13) represent 16 coupled hyperbolic-elliptic nonlinear partial differential equations, only 10 of which are independent
- Involve second derivatives of the metric

[3] It is customary to set $8\pi G = 1 = c$.

EFEs for Weak Gravitational Fields

- In the absence of gravity, spacetime is flat
- In a weak gravitational field (WGF), spacetime is nearly flat $\rightarrow$ a manifold on which coordinates exist such that metric components can be written as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\ , \quad |h_{\alpha\beta}| \ll 1 \tag{2.16}$$

- Nearly Lorentzian coordinates/metric
- Recall that in SR, the Lorentz transformation matrix for a boost of velocity v in the x-direction is given by

$$\Lambda^{\alpha'}{}_{\beta} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ , \quad \gamma = \frac{1}{\sqrt{1-(v/c)^2}}$$

- The background Lorentz transformation in WGF can, therefore, be written as

$$x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^{\beta} \tag{2.17}$$

- Transforming the metric tensor gives

$$\begin{aligned} g_{\alpha'\beta'} &= \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'} g_{\mu\nu} = \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}_{\beta'}\eta_{\mu\nu} + \Lambda^{\mu}{}_{\alpha'}\Lambda^{\nu}{}_{\beta'} h_{\mu\nu} \\ &= \eta_{\alpha'\beta'} + h_{\alpha'\beta'} \end{aligned} \tag{2.18}$$

- The gauge transformation

$$x^{\alpha'} \rightarrow x^{\alpha} + \xi^{\alpha}(x^{\beta}) \ , \quad |\xi^{\alpha}{}_{,\beta}| \ll 1 \tag{2.19}$$

yields:

$$\Lambda^{\alpha'}{}_{\beta} = x^{\alpha'}_{,\beta} = \delta^{\alpha}{}_{\beta} + \xi^{\alpha}{}_{,\beta} \tag{2.20}$$

$$\Lambda^{\alpha}{}_{\beta'} = x^{\alpha}_{,\beta'} = \delta^{\alpha}{}_{\beta} - \xi^{\alpha}{}_{,\beta} + O(|\xi^{\alpha}{}_{,\beta}|^2) \tag{2.21}$$

and therefore, to first order,

$$g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{2.22}$$

- Thus, redefine $h_{\alpha\beta}$:

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{2.23}$$

- To first order in $h_{\alpha\beta}$, one can see from (1.85) and (2.16) that:

$$R_{\alpha\beta\delta\theta} = \frac{1}{2}\left[h_{\alpha\theta,\beta\delta} + h_{\beta\delta,\alpha\theta} - h_{\alpha\delta,\beta\theta} - h_{\beta\theta,\alpha\delta}\right] \tag{2.24}$$

[Ex] Show that the Einstein tensor is given by

$$G_{\alpha\beta} = -\frac{1}{2}\left[\bar{h}_{\alpha\beta,\delta}{}^{,\delta} + \eta_{\alpha\beta}\bar{h}_{\delta\theta}{}^{,\delta\theta} - \bar{h}_{\alpha\delta,\beta}{}^{,\delta} - \bar{h}_{\beta\delta,\alpha}{}^{,\delta} + O(h^2_{\alpha\beta})\right] \tag{2.25}$$

where

$$h \equiv h^{\alpha}{}_{\alpha} \ , \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h \tag{2.26}$$

- We can choose a gauge such that

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 \quad \text{Lorentz/harmonic/de Donder gauge condition} \tag{2.27}$$

- In this gauge, (2.25) becomes

$$G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta} \tag{2.28}$$

- Thus the Einstein weak-field equations (EWFEs) read

$$\Box\bar{h}^{\alpha\beta} = -16\pi G T^{\alpha\beta} \tag{2.29}$$

and describe the field equations of linearized gravity.

The Newtonian Limit

- Einstein's GR is a generalization of Newtonian gravitation (NG). One therefore expects that in the appropriate limit, GR reduces to NG.
- Newton's two laws of gravitation $\longleftrightarrow$ Einstein's GR:

$$\mathbf{F} = m\mathbf{a}\ , \quad \mathbf{F} = -m\nabla\Phi \longleftrightarrow \text{GE} \tag{2.30}$$

$$\nabla^2\Phi = 4\pi G\mu \longleftrightarrow \text{EFEs} \tag{2.31}$$

- In the Newtonian limit
  - gravity is too weak (gravitational potential energy of a particle is much less than its rest-mass energy), i.e.,

    $$\Phi \ll 1 \tag{2.32}$$

    to produce velocities near the speed of light[4], i.e.,

    $$v/c \ll 1 \tag{2.33}$$

  - Time derivatives are much smaller than spatial derivatives
  - The [Newtonian] potential Φ completely determines the metric:

$$ds^2 = -(1 + 2\Phi/c^2)dt^2 + (1 - 2\Phi/c^2)\left[dx^2 + dy^2 + dz^2\right] \tag{2.34}$$

- From (2.34)

$$g_{00} = -(1 + 2\Phi/c^2) \tag{2.35}$$

[4] In this subsection, let's keep c and $8\pi G$ for the sake of dimensionality.

- In the near-Minkowski spacetime, the affine connection is given by

$$\Gamma^{\theta}_{\alpha\beta} = \frac{1}{2}\eta^{\theta\delta}\left[h_{\delta\beta,\alpha} + h_{\alpha\delta,\beta} - h_{\alpha\beta,\delta}\right] \tag{2.36}$$

- The GE (equation of motion for a freely falling particle):

$$\frac{d^2x^\theta}{d\tau^2} + \Gamma^{\theta}{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{2.37}$$

- For a nonrelativistic particle, $\tau \approx t$:

$$\frac{d^2x^\theta}{dt^2} + \Gamma^{\theta}{}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0 \tag{2.38}$$

- Neglect terms like $\Gamma^{\theta}{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$ since

$$\frac{dx^i}{dt} = O(h^{\alpha\beta}) \tag{2.39}$$

- Rewrite the GE (2.38):

$$\frac{d^2x^\theta}{dt^2} + \Gamma^\theta{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} = 0 \tag{2.40}$$

- Split (2.40) into spatial and temporal equations (using $\frac{dx^0}{dt} = c$):

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} = 0 \tag{2.41}$$

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i{}_{00} \tag{2.42}$$

- One can show that

$$\Gamma^i_{00} = \frac{1}{2}\left[2h_{i0,0} - h_{00,i}\right] \approx -\tfrac{1}{2}h_{00,i} \tag{2.43}$$

- Thus a further simplication of the spatial component of the GE gives

$$\frac{d^2x^i}{dt^2} = \tfrac{1}{2}c^2h_{00,i} = \tfrac{1}{2}c^2\nabla_i h_{00} \tag{2.44}$$

- Cf. this with Newton's Second law:

$$\frac{d^2x^i}{dt^2} = -\nabla_i\Phi \implies h_{00} = -2\Phi/c^2 \tag{2.45}$$

- $$\therefore ds^2 = -(1+2\Phi/c^2)dt^2 + (1-2\Phi/c^2)\left[dx^2 + dy^2 + dz^2\right] \tag{2.46}$$

- [Ex] Show that one can rewrite the EFEs (2.13) as

$$R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \tfrac{1}{2} T g_{\alpha\beta} \right) \tag{2.47}$$

  and that for a perfect fluid,

$$R_{\alpha\beta} = \kappa \left(\mu + p/c^2 \right) u_\alpha u_\beta + \tfrac{1}{2}\kappa \left(\mu - p/c^2 \right) c^2 g_{\alpha\beta} \tag{2.48}$$

- Newtonian limit:

$$\mu \gg p/c^2 \tag{2.49}$$

- For dust, $p = 0$

$$R_{\alpha\beta} = \kappa\mu u_\alpha u_\beta + \tfrac{1}{2}\kappa c^2 \mu g_{\alpha\beta} \tag{2.50}$$

- To first order in $h_{\alpha\beta}$, the (00) - component of $R_{\alpha\beta}$ is

$$R_{00} = \kappa c^2 \mu + \tfrac{1}{2}\kappa c^2 \mu(-1) = \tfrac{1}{2}\kappa c^2 \mu \tag{2.51}$$

- From the def. (1.90), one can see that to first order in $h_{\alpha\beta}$

$$R_{00} = {\Gamma^\delta}_{00,\delta} - {\Gamma^\delta}_{0\delta,0} \approx {\Gamma^i}_{00,i} = -\tfrac{1}{2} h_{00,ii} = \tfrac{1}{2}\kappa c^2 \mu \tag{2.52}$$

- Cf. this with (2.45):

$$\nabla^2 \Phi = \tfrac{1}{2} k c^4 \mu \tag{2.53}$$

  and this with (2.31) to obtain

$$\kappa = \frac{8\pi G}{c^4} \tag{2.54}$$

The Far-field Limit

- Assume the source of the field $T^{\alpha\beta}$ is stationary
- Assume also that, far away, $h_{\alpha\beta}$ is stationary

$$\lim_{r\to\infty} g_{\alpha\beta} = \eta_{\alpha\beta} \longrightarrow \text{asymptotically flat} \tag{2.55}$$

- Far from source, (2.29) becomes

$$\Box \bar{h}^{\alpha\beta} = 0 \tag{2.56}$$

  and has the stationary-source solution

$$\bar{h}^{\alpha\beta} = \frac{A^{\alpha\beta}}{r} + O(1/r^2) \quad \text{(for some constant } A^{\alpha\beta}\text{)} \tag{2.57}$$

- From the gauge condition (2.27):

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 = \bar{h}^{\alpha i}{}_{,i} = -\frac{A^{\alpha i} n_i}{r^2} + O(1/r^3) \quad \text{(Why?)} \tag{2.58}$$

  where $n_i \equiv x_i/r$ is the unit radial normal
- Eq. (2.58) implies

$$A^{\alpha i} = 0 \;\forall \alpha, i \tag{2.59}$$

- Eq. (2.57) $\implies$ $\bar{h}^{00}$ is the only non-vanishing part (Why?)

- Now, using (2.26) and (2.45), we can define the Newtonian potential for the far field of any stationary relativistic source:

$$\Phi_{ffsrs} \equiv -\tfrac{1}{4}\bar{h}^{00}_{ff} \tag{2.60}$$

- For a Newtonian source, the far-field potential is

$$\Phi_{nff} = -\frac{M}{r} + O(1/r^2) \tag{2.61}$$

- Rename the constant A^{00} of Eq. (2.57) to be $4M$ so that

$$\Phi_{ffsrs} = -\frac{M}{r} \tag{2.62}$$

- $M \longrightarrow$ total mass of the relativistic source
- Any small body (e.g, a planet) falling into the relativistic source's gravitational field but stays away from it will follow the geodesics of the metric

$$ds^2 = -\left[1 - \frac{2M}{r} + O(1/r^2)\right] dt^2 + \left[1 + \frac{2M}{r} + O(1/r^2)\right] \left(dx^2 + dy^2 + dz^2\right) \tag{2.63}$$

- If we relax the 'stationary-source' assumption, then a source which changes with time can emit gravitational waves. (Which equation should we be looking at???)

Spherically Symmetric Spacetimes

- Many astrophysical objects are nearly spherical
- Minkowski space in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.64}$$

- Spherically symmetric spacetimes: simplest possible form of metric

$$ds^2 = -g_{tt}dt^2 + 2g_{tr}drdt + g_{rr}dr^2 + r^2 d\Omega^2 \tag{2.65}$$

- g_{tt} , g_{tr} , g_{rr} functions of r, t
- Static spacetime $\rightarrow$ spacetime with a time coordinate t such that
  - all metric components are independent of t
  - geometry is unchanged by time-reversal, $t \rightarrow -t$
- Stationary spacetime $\rightarrow$ spacetime with the first, but not necessarily the second property
- The second property has the following implications: the coordinate transformation

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\psi(r)}dr^2 + r^2 d\Omega^2 \tag{2.66}$$

- Stars $\rightarrow$ bounded systems $\implies$ far from the star, spacetime is flat
- Asymptotic regularity (boundary) conditions on EFEs:

$$\lim_{r\rightarrow\infty} \varphi(r) = 0 = \lim_{r\rightarrow\infty} \psi(r) \tag{2.67}$$

Realistic Stars and Their Evolution

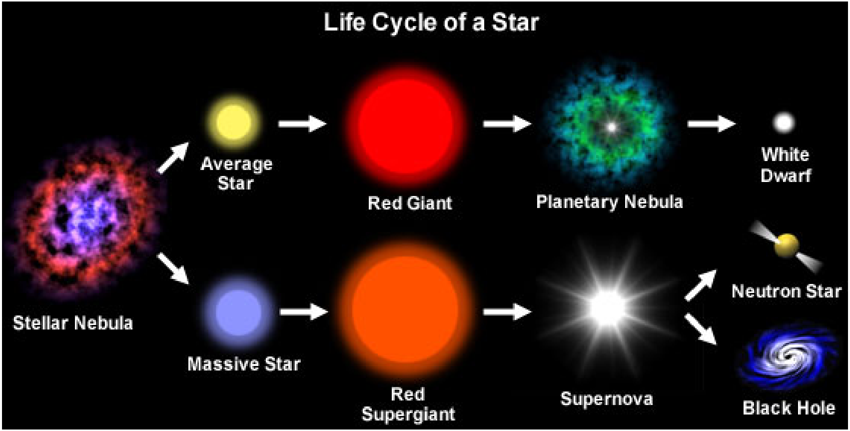


Figure 2.1: Stellar Evolution. [Credit: Sciology.org]

Propagation of GWs

- Gravitational waves (GWs) are produced in a region of spacetime where the gravitational field is weak but not stationary
- *E.g.*, far from a fully relativistic source undergoing rapid changes (such as the Big Bang or gravitational collapse to a blackhole) that took place long enough ago for the disturbances produced by the changes to reach distant regions
- We have seen the EWFEs in Eq. (2.29), which in vacuum read:

$$\Box \bar{h}^{\alpha\beta} = 0 = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\alpha\beta} \tag{2.68}$$

- This is the 3-D wave equation and has a plane-wave solution of the form

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha} \tag{2.69}$$

where k_α and $A^{\alpha\beta}$ are the constant (amplitude) components of some vector and tensor, respectively

- Eq. (2.69) is a wavelike solution; $\bar{h}^{\alpha\beta}$ is const if

$$k_\alpha x^\alpha = k_0 t + \mathbf{k}.\mathbf{x} = const \tag{2.70}$$

$$k^0 \equiv \omega \rightarrow \text{frequency of wave} \tag{2.71}$$

$$\mathbf{k} \rightarrow k^i \tag{2.72}$$

- Dispersion relation for the wave follows from the nullity of k^α:

$$\omega^2 = |\mathbf{k}|^2 \tag{2.73}$$

- $\rightarrow$ wave's phase and group velocities both equal 1!

Cosmology

- studies about the large scale structure, origin, evolution and the ultimate fate of the Universe as a whole.
- answers (attempts to answer) crucial questions about the properties of the Universe such as

  Size: how large is the Universe?
  Shape: what shape does it have?
  Age: how old is it?
  Evolution: how did it form? What will happen to it in the future?
- Modern cosmology started with Einstein's publication of the General Relativity Theory (GR) ca. 1915
- With the advent of Einstein's GR describing space-time as a dynamical continuum evolving according to the local content of energy and momentum came the idea of an expanding universe
- Solutions to Einstein's equations corresponding to expanding universes were found by de Sitter (ca. 1917), Friedmann (1922) and Lemaître (1927) but the general opinion, including that of Einstein himself, was that the real Universe[5] should be static

[5] Note: Until the early 20th century, our conception of the Universe was limited to that of the Milky Way.

Why Gravity?

- Four fundamental forces in nature; why care only about gravity?

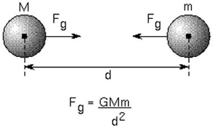


Figure 3.1: Newton's universal gravitation (1687). Gravity as force-at-a-distance. [Credit: physics.stackexchange.com]

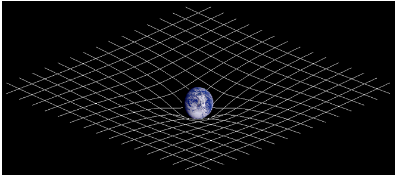


Figure 3.2: Einstein's GR (1915). Gravity as curvature of spacetime. "Matter tells spacetime how to curve, spacetime tells matter how to move."

Interaction	Current Theory	Mediators	Relative Strength	Long-Distance Behavior	Range (m)
Strong	Quantum chromodynamics (QCD)	gluons	10^{38}	1	10^{-15}
Electromagnetic	Quantum electrodynamics (QED)	photons	10^{36}	$\frac{1}{r^2}$	∞
Weak	Electroweak Theory	W and Z bosons	10^{25}	$\frac{d}{dr}\left(\frac{\exp(-m_{W,Z}r)}{r}\right)$	10^{-18}
Gravitation	General Relativity (GR)	gravitons (hypothetical)	1	$\frac{1}{r^2}$	∞

Figure 3.3: The fundamental forces of nature.

The Expanding Universe

- Einstein's field equations of GR predict an expanding universe
- Einstein added the cosmological constant to make the universe static
- Observations confirmed an expanding universe (1920s)
- Einstein committed " the biggest blunder of my scientific career"

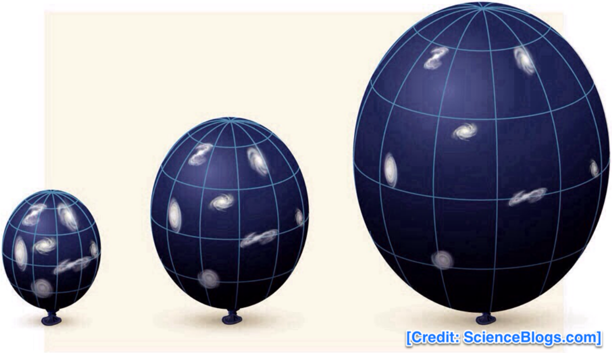


Figure 3.4: The expanding Universe.

The Concordance Cosmological Model

- Based on the current cosmological paradigm, the visbile Universe is
  - homogeneous: all regions of space look alike, no preferred positions
  - isotropic: no preferred directions

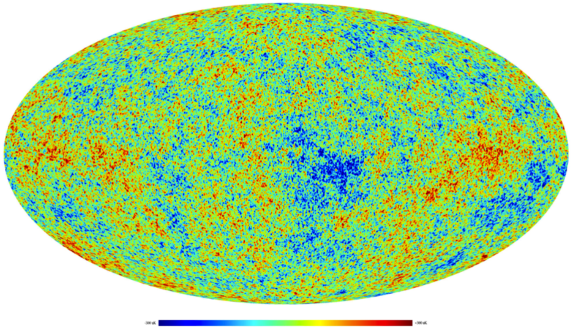


Figure 3.5: The Cosmic Microwave Background (CMB). Today, $T \sim 2.726K$, $\frac{\delta T}{T} \sim 10^{-5}$. [Credit: www.esa.int]

- Based on the Friedmann-Lemaître-Robertson-Walker (FLRW) metric
- Globally homogenous, isotropically expanding/contracting spacetime geometry

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + f^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{3.1}$$

- $a(t) \equiv$ the cosmological scale factor, tells us the relative size of the Universe.

$$f(r) = \begin{cases} \sin(r) & \text{for } k = +1, \\ r & \text{for } k = 0, \\ \sinh(r) & \text{for } k = -1. \end{cases}$$

- Normalized 4-velocity of fundamental observers, worldlines with tangent vector

$$u^a = \frac{dx^a}{dt} \tag{3.2}$$

- The $t = constant$ space sections are surfaces of homogeneity and have maximal symmetry: they are 3-spaces of constant curvature

$$K = \frac{k}{a^2(t)}\ , \tag{3.3}$$

where k is the sign of K and hence takes the values $-1, 0$ or $+1$ depending on whether the Universe is open, flat or closed, respectively.

- The rate of cosmic expansion at any time t is characterized by the Hubble parameter

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{3.4}$$

- The redshift z of an object emitting a wavelength λ_e (and a frequency ν_e) and observed with wavelength λ_o (and a corresponding frequency ν_o) is defined to be the fractional Doppler shift of its emitted light (photons) due to its radial motion:

$$z \equiv \frac{\lambda_o}{\lambda_e} - 1 = \frac{\nu_e}{\nu_o} - 1 = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \qquad (\text{where } \beta \equiv \frac{v}{c}) \tag{3.5}$$

$$z = \frac{a_0}{a(t)} - 1 \tag{3.6}$$

[Ex] Use Taylor expansion to show that, for $z \ll 1$,

Hubble's Law

$$v = H_0 D \tag{3.7}$$

with the 'distance' ('naive Hubble distance') D to a galaxy and v its velocity of recession. H_0 is Hubble's original proportionality constant, today known as the 'Hubble constant'.

The Hubble Constant

- One of the most important cosmological parameters
- SI unit (show!) is s^{-1}, but values are usually quoted in $km/s/Mpc$
- 1 $parsec \simeq 3.086 \times 10^{16}$ $meters \simeq 3.262$ $ly \rightarrow$ distance at which one astronomical unit subtends an angle of one arcsecond

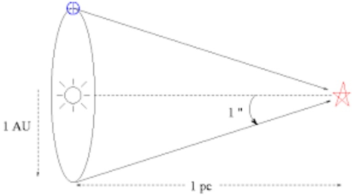


Figure 3.6: [Credit: spiff.rit.edu]

- $km/s/Mpc \rightarrow$ the speed in km/s of a galaxy 1Mpc away
- Hubble's initial value: $H_0 \sim 500$ $km/s/Mpc \sim 160 km/s$ per $million$ $light - years$
- Common notation: $H_0 = 100h$ $km/s/Mpc$, $h \rightarrow$ uncertainty parameter
- H_0 sets a time scale H_0^{-1}, and together with the matter and energy content of the Universe, sets the age of the Universe.

Date published	Hubble constant (km/s)/Mpc	Observer
2016-07-13	$67.6^{+0.7}_{-0.6}$	SDSS-III Baryon Oscillation Spectroscopic Survey
2016-05-17	73.00 ± 1.75	Hubble Space Telescope
2013-03-21	67.80 ± 0.77	Planck Mission
2012-12-20	69.32 ± 0.80	WMAP (9-years)
2010	$70.4^{+1.3}_{-1.4}$	WMAP (7-years), combined with other measurements.
2010	71.0 ± 2.5	WMAP only (7-years).
2009-02	70.1 ± 1.3	WMAP (5-years). combined with other measurements.
2009-02	$71.9^{+2.6}_{-2.7}$	WMAP only (5-years)
2007	$70.4^{+1.5}_{-1.6}$	WMAP (3-years)
2006-08	$77.6^{+14.9}_{-12.5}$	Chandra X-ray Observatory
2001-05	72 ± 8	Hubble Space Telescope
prior to 1996	50–90 (est.)	
1958	75 (est.)	Allan Sandage

Figure 3.7: The value of the Hubble constant. [Credit: WIKI]

Dynamics of Cosmic Expansion

- The EFEs show the effect of matter on space-time curvature.
- Matter and energy are sourced by the Energy-Momentum Tensor (EMT) T_{ab} , which for perfect-fluid FLRW universes is given by

$$T_{ab} = \mu u_a u_b + p h_{ab} \ , \tag{3.8}$$

  where

$$h_{ab} = g_{ab} + u_a u_b \tag{3.9}$$

  is the projection tensor into the tangent 3-spaces orthogonal to u^a.
- The energy density and the pressure terms $\mu(t)$ and $p(t)$ are the time-like and space-like eigenvalues of T_{ab}, respectively.
- The evolution of the energy density gives the conservation equation

$$T^{ab}{}_{;b} = 0 \Leftrightarrow \dot{\mu} + (\mu + p)\Theta = 0 \ , \tag{3.10}$$

  where $\Theta \equiv 3H$.
- Controls the density of matter as the Universe expands

- Equation of state

$$w = \frac{p}{\mu} \tag{3.11}$$

- For example, Cold Dark Matter (**dust**) is pressureless and hence $w_d = 0$ whereas radiation has

$$p_r = \mu_r/3 \Leftrightarrow w_r = 1/3 \tag{3.12}$$

- Photon energy density

$$\mu_r \propto T^4 \tag{3.13}$$

[Ex] Using the conservation equation (3.10), show that

$$\mu_d \propto a^{-3}, \quad \mu_r \propto a^{-4}, \quad T \propto a^{-1}. \tag{3.14}$$

- We can also think of the cosmological constant Λ as a fluid such that

$$p_\Lambda = -\mu_\Lambda \tag{3.15}$$

and hence with a corresponding equation of state $w_\Lambda = -1$.
[Ex] How does the energy density of Λ grow/decay with the scale factor a?

The Friedmann Equation

- Shows how matter directly causes a curvature of 3-spaces:

$$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{\mu}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{3.16}$$

- Controls the expansion of the Universe
- Note here that μ is the total energy density of all kind of matter in the Universe. For example, if we include baryons (b), radiation (r), Cold Dark Matter (CDM) and neutrinos (ν), then

$$\mu = \mu_b + \mu_r + \mu_{CDM} + \mu_\nu \ , \tag{3.17}$$

  and this requires the specification of the equation of state w_i for each component i of matter
- Since CDM is by far the dominant component in a baryon-CDM mixture, it is customary to approximate the mixture as pressureless *dust*:

$$\mu_d \equiv \mu_b + \mu_{CDM} \tag{3.18}$$

- It is also customary to combine dust and radiation fluids as *standard matter* and write:

$$\mu_m \equiv \mu_d + \mu_r \tag{3.19}$$

[Ex] Show that the Friedmann equation can be written in a more compact form as

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1 \tag{3.20}$$

where Ω_i is a normalized dimensionless density parameter for the ith fluid component.

The Raychaudhuri Equation

- Gives the basic evolution equation for the scale factor $a(t)$

$$3\frac{\ddot{a}}{a} = -\tfrac{1}{2}(\mu + 3p) + \Lambda\ , \tag{3.21}$$

  and hence the basic equation of gravitational interactions and the basis of singularity theorems in GR
- For ordinary matter the *Strong Energy Condition* (SEC) imposes a positive gravitational mass density according to

$$\mu + 3p > 0 \Leftrightarrow w > -1/3\ . \tag{3.22}$$

- Ordinary matter will tend to cause the Universe to decelerate ($\ddot{a} < 0$) whereas a positive cosmological constant, according to Eq. (3.21), causes an accelerated expansion ($\ddot{a} > 0$).
- Deceleration parameter

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} \tag{3.23}$$

Initial Conditions

- Ensure the existence of unique cosmological solutions
- At an arbitrary time t_0 (say, today) [6], initial data for such solutions consists of :

  The Hubble constant

$$H_0 = \left[\frac{\dot{a}}{a}\right]|_{t_0} = 100h \ \ \text{km/sec/Mpc} \tag{3.24}$$

  A dimensionless normalized density parameter

$$\Omega_{i0} = \frac{\mu_{i0}}{3H^2} \tag{3.25}$$

  for each type of matter present;

  For a non-vanishing cosmological constant , i.e., $\Lambda \neq 0$, either the fractional energy density

$$\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2} \tag{3.26}$$

  The dimensionless **deceleration parameter**

$$q_0 = -\left[\frac{\ddot{a}}{a}\right]|_{t_0} H_0^{-2} \tag{3.27}$$

- Determine unique corresponding cosmic history

[6] Here, however, a subscript 0, unless otherwise indicated, will refer to the present epoch.

- If the pressure term p is negligible relative to the matter term μ in (3.21), then we get (show!)

$$q_0 = \frac{1}{2}\Omega_{m0} - \Omega_{\Lambda 0} \,. \tag{3.28}$$

- (3.28) $\implies$ a dominant cosmological constant ($\Omega_{\Lambda 0} > \Omega_{m0}$) causes an accelerated cosmic expansion ($q_0 < 0$).
- Matter can cause deceleration ($q > 0$) of the expansion if $\Omega_{m0} > 2\Omega_{\Lambda 0}$; in particular,

$$\Lambda = 0 \implies q_0 = \frac{1}{2}\Omega_{m0} \implies \text{deceleration} \tag{3.29}$$

- The spatial curvature

$$K_0 = \frac{k}{a_0^2} = H_0^2(\Omega_0 - 1) \tag{3.30}$$

can be obtained by evaluating the Friedmann equation (3.16) at the present time t_0. The Universe is said to be

$$\begin{cases} \text{open} & \text{if } K_0 < 0 \implies \Omega_0 < 1 \\ \text{flat} & \text{if } K_0 = 0 \implies \Omega_0 = 1 \\ \text{closed} & \text{if } K_0 > 0 \implies \Omega_0 > 1 \end{cases}$$

where $\Omega_0 \equiv \Omega_{d0} + \Omega_{\Lambda 0}$

Solutions

- The FLRW models are the most widely explored cosmological models
  - Extremely simple geometry
  - Ever-increasing accuracy of supporting observational data
- Defining $\rho_D \equiv \mu_{d0} a_0^3$ and $\rho_R \equiv \mu_{r0} a_0^4$ such that $\dot{\rho}_D = 0$ and $\dot{\rho}_R = 0$, one can rewrite Eq. (3.16) for dust and non-interacting radiation as

$$3\frac{\dot{a}^2}{a^2} = \frac{\rho_D}{a^3} + \frac{\rho_R}{a^4} + \Lambda - 3\frac{k}{a^2}\ . \tag{3.31}$$

For $(\Lambda = 0)$

- The Universe starts off at a very dense initial state, where its energy density and curvature tend to infinity
- Future fate depends on the value of the spatial curvature, or equivalently the density parameter Ω_0.
- Expands forever if $k = 0 \Leftrightarrow \Omega_0 = 1$ or $k < 0 \Leftrightarrow \Omega_0 < 1$
- Collapses to a future singularity if $k > 0 \Leftrightarrow \Omega_0 > 1$
- $\Omega_0 = 1$ corresponds to the critical density μ_{crit} separating $\Lambda = 0$ FLRW models that recollapse in the future from those that expand forever, and Ω_0 is just the ratio of the matter density to this critical density:

$$\Omega_{crit} = 1 \Leftrightarrow \mu_{crit} = 3H_0^2 \Rightarrow \Omega_0 = \frac{\mu_0}{3H_0^2} = \frac{\mu_0}{\mu_{crit}} \tag{3.32}$$

For $\Lambda < 0$

- All solutions start at a singularity and recollapse.

For $\Lambda > 0$ scenarios:

- If $k = 0$ or $k = -1$, all solutions start at a singularity and expand forever.
- If $k = +1$, there can be
  - models with a singular start, either expanding forever or collapsing to a future singularity
  - a static solution (the Einstein static universe)
  - models asymptotic to Einstein static state in either the future or the past.
- Models with $k = +1$ can bounce (collapsing from infinity to a minimum radius and re-expanding).

Some very specific models with simple expanding solutions

- The Einstein-de Sitter model
  - This is the simplest ($p = 0, \Lambda = 0, k = 0$) ($\Rightarrow \Omega_0 = 1$) expanding non-empty solution

$$a(t) = Ct^{2/3} \quad (\text{show!}) \,, \tag{3.33}$$

    where C is an integration constant.
  - This solution starts from a singular state at time $t = 0$.
  - The age of the Universe in this model (the proper time since the start of the Universe) when the Hubble constant takes the value H_0 is (show!)

$$\tau_0 = \frac{2}{3H_0} \tag{3.34}$$

- The Milne model
  - Characterized by $(\mu = p = 0, \Lambda = 0, k = -1) \Rightarrow \Omega_0 = 0$ and represents a linearly expanding empty solution

$$a(t) = Ct \quad \text{(show!)} \ , \tag{3.35}$$

    in a flat spacetime as seen by a uniformly expanding set of observers, singular at $t = 0$.
  - The age of the Universe in this model is given by (show!)

$$\tau_0 = \frac{1}{H_0} \tag{3.36}$$

- The de Sitter universe
  - Characterized by $(\mu = p = 0, \Lambda \neq 0, k = 0) \Rightarrow \Omega_0 = 0$
  - The Universe is in a steady state of expansion without matter, the empty solution given by

$$a(t) = Ce^{Ht} \ , \quad \text{(show!)} \tag{3.37}$$

    where C and H are constants
  - The Universe expands at a constant rate in this model, and hence there is no start and its age is infinite (show!)

[Ex] Show that a de Sitter universe with $k = +1$ has the bounce solution given by

$$a(t) = a_0 \cosh(Ht) \tag{3.38}$$

whereas that with $k = -1$ has the singular start solution

$$a(t) = a_0 \sinh(Ht) \ . \tag{3.39}$$

Cosmography

- Science of measuring the ‘distance’ between two observed cosmological objects or events
- Challenging: lack of accurate cosmological data + cosmic expansion
- Use direct observable such as the luminosity of a quasar, the redshift of a galaxy, or the angular size of the CMB power spectrum acoustic peaks to indirectly measure another quantity not directly observable, but mathematically calculable, such as the comoving coordinates of the quasar or the galaxy.

- The Hubble time given by $\tau_H = \frac{1}{H_0}$ is the time taken for light to traverse a Hubble distance

$$D_H = \frac{c}{H_0} \ . \tag{3.40}$$

- Based on current H_0 estimates coming from:
  - recession velocities of objects around us

$$H_0 = 73 \pm 2.4 \mathrm{km} s^{-1} Mpc^{-1} \tag{3.41}$$

  - the 9-year WMAP analysis

$$H_0 = 68.65 \pm 0.93 \mathrm{km} s^{-1} Mpc^{-1} \tag{3.42}$$

  - Planck satellite's most recent results

$$H_0 = 67.3 \pm 1.2 \mathrm{km} s^{-1} Mpc^{-1} \tag{3.43}$$

we get

$$\tau_H \simeq 1.2 - 1.5 \times 10^{10} \mathrm{years} \tag{3.44}$$
$$D_H \simeq 1.2 - 1.5 \times 10^{26} \mathrm{m} \simeq 3700 - 4700 \mathrm{Mpc} \tag{3.45}$$

- Normalized to the geometric units:

$$\tau_H = D_H = c = 1 \tag{3.46}$$

Redshift

- The redshift z of an object emitting a wavelength λ_e (and a frequency ν_e) and observed with wavelength λ_o (and a corresponding frequency ν_o) is defined to be the fractional Doppler shift of its emitted light (photons) due to its radial motion:

$$z \equiv \frac{\lambda_o}{\lambda_e} - 1 = \frac{\nu_e}{\nu_o} - 1 \tag{3.47}$$

- The redshift of a moving object in an expanding universe can be given by

$$1 + z = (1 + z_c)(1 + z_v), \tag{3.48}$$

where z_v is the redshift due to the local peculiar motion of the object whereas z_c is the cosmological redshift due to the expansion of the Universe given in terms of the scale factor as

$$1 + z_c = \frac{a(t_0)}{a(t_e)}. \tag{3.49}$$

- For comoving objects, we have $z_v = 0$ from which follows $z_c = z$.

Shortcomings of the Standard Model

- The Hot Big Bang model is by far the most successful cosmological paradigm in explaining
  - the expansion of the Universe
  - the origin of the CMB
  - the synthesis of light elements
  - the formation of galaxies and large-scale structure

- It leaves many serious puzzles unanswered, some of them originating in the early Universe, some appearing in the late-time Universe
  - The Horizon Problem : why do causally disconnected regions in the Universe share similar physical properties such as temperature?
  - The Flatness Problem : fine-tuning of the initial conditions which would otherwise have greatly affected the geometry and expansion history of the Universe
  - The Structure Problem (Smoothness Problem/Homogneity Problem) : what brought about the matter clumping that finally led to cosmic structures like galaxies and clusters?
  - The Anti-Matter Problem (Baryon Asymetry Problem) : at high enough temperatures $kT \geq m_p c^2$, where m_p is the mass of a proton, there are roughly equal numbers of photons (γ), protons (p) and antiprotons ($\bar{p}$) in equilibrium, whereas the ratios today stand at $N_p/N_\gamma \sim 10^{-9}$ and $N_{\bar{p}}/N_p \sim 0$. Since baryon number is a conserved quantity, it would then necessarily imply that $N_p/N_{\bar{p}} = 1 + O(10^{-9})$ during baryogenesis
  - The Magnetic-Monopole Problem (Exotic Relics Problem) : the Universe were very hot at early times $\rightarrow$ a large number of heavy, stable magnetic monopoles would be produced, and should be detected observationally
  - Rotational curves of galaxies $\rightarrow$ dark matter
  - Cosmic speed -up $\rightarrow$ Dark Energy

The Cosmological Crisis

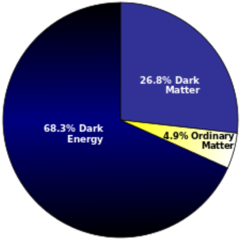


Figure 3.8: The cosmic pie crisis.

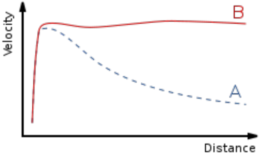


Figure 3.9: Rotation curves of galaxies: predicted (A) and observed (B).

Dark Matter and Dark Energy

- Dark matter and dark energy collectively account for the dark side of the Universe, i.e., the part of the Universe that is not in the luminous (baryonic) form.
- If the standard Big Bang model is the correct gravitational theory of cosmology, then the matter-energy contents of the Universe should comprise the following:
  - $\Omega_m \sim 0.315$, i.e., $\sim 31.5\%$ of the total energy of the Universe exists in the form of non-relativistic matter, of which only a tiny fraction ($\Omega_b \sim 0.049$) is known to exist in baryonic matter form, whereas the remaining ($\sim 85\%$ of) matter is not as yet properly understood and hence is thought to exist in the form of *dark matter* (=CDM+HDM)
- Dark energy dominates the Universe with $\Omega_\Lambda \sim 0.685$
  - Phenomenological, not predicted from the Big Bang cosmology
  - Little understood dynamical nature
- Einstein's Λ brought back to life?
  - The cosmological constant problem: huge ($\sim 10^{120}$ orders of magnitude) discrepancy between the "observed" value of Λ responsible for cosmic acceleration and that predicted by quantum field theoretic arguments for the energy of the quantum vacuum at Planckian scales
  - The coincidence problem: The total fractional energy density is close to 1.0 at the present time when we are here to observe it, after about 13.82 billion years of expansion when it was always greater than 1.0. Since Ω_Λ is the only constant component, it is natural to be curious about why Λ is so finely tuned as to be dominant only now
- One rather philosophical solution to this is the *Anthropic Principle*: we see the Universe the way it is because we exist

Inflation

- Proposed to solve the horizon, flatness, structure and magnetic monopole problems
- Special epoch of exponential expansion moments after the Big Bang ($\sim 10^{-36}s - \sim 10^{-32}s$)
- Decreasing comoving Hubble radius, i.e.,

$$\frac{d(1/aH)}{dt} < 0 \Leftrightarrow \ddot{a} > 0. \tag{3.50}$$

- Hubble radius swallowed by the outpacing accelerated growth of the expansion
- All physical conditions become correlated on scales much larger than the Hubble radius, thus *smoothing* out the primordial matter fluctuations along the way

- Assumption: at some point in the early universe, the matter energy density was dominated by some form of matter, called a *scalar field* ϕ with a negative pressure
- In the absence of the cosmological constant, the Raychaudhuri equation (3.21) reduces to

$$\ddot{a} = -\frac{1}{6}(\mu + 3p)a \implies p < -\frac{1}{3}\mu \tag{3.51}$$

- The Friedmann equation (3.16) reads

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\mu - \frac{k}{a^2}. \tag{3.52}$$

- Curvature term becomes negligible since the scale factor must increase faster than $a(t) \propto t$

Inhomogeneous cosmologies

- The Concordance Model (Λ*CDM Cosmology*) is based on large-scale homogeneity and isotropy
- Largely motivated by the observed near isotropy of the CMB and the Cosmological/Copernican Principle
- FLRW geometry may not be the right geometry for all scales
- If the Universe is inherently inhomogeneous, only inhomogeneous cosmological models, such as the Lemaître-Tolman-Bondi (LTB), Szekeres and Kantowski-Sachs models can best describe it

Modified/Generalized/Alternative Theories of Gravity

- Theories of Gravity with Extra Fields
  - Scalar-Tensor theories
    - * Brans-Dicke theories (BD)
  - Einstein-Æther theories
    - * Modified Newtonian Dynamics (MOND)
  - Bimetric theories
  - Tensor-Vector-Scalar theories (TeVeS).
- Higher-dimensional Theories of Gravity
  - Kaluza-Klein (KK) theories
  - Braneworld models
  - Randall-Sundrum (RS) models
  - Dvali-Gabadadze-Porrati (DGP) gravity
  - (Einstein)Gauß-Bonnet (GB) gravity.
- Higher-derivative Theories of Gravity
  - Theories with Ricci and Riemann curvatures in the action
  - Hořava-Lifshitz gravity
  - Galileons
  - $f(R)$ theories: theories with a generic function of the Ricci scalar in the action Lagrangian.

Table 1: Summary of Gravitational Models and their Lagrangians:

Model	Action	Matter Lagrangian
GR	$\frac{1}{2}\int d^4x\sqrt{-g}\,(R-2\Lambda)$	$\mathcal{L}_m(g_{ab},\psi)$
BD	$\frac{1}{2}\int d^4x\sqrt{-g}\left(\phi R-\frac{\omega\,\partial_a\phi\partial^a\phi}{\phi}\right)$	$\mathcal{L}_m(g_{ab},\psi)$
MOND	$\frac{1}{2}\int d^4x\sqrt{-g}\left(R+2\mathcal{L}(g^{ab},A^b)\right)$	$\mathcal{L}_m(g^{ab},\psi)$
TeVeS	$\frac{1}{2}\int d^4x\left(\mathcal{L}_g+\mathcal{L}_s+\mathcal{L}_v\right)$	$\mathcal{L}_m(g_{ab},\psi)$
$f(R)$	$\frac{1}{2}\int d^4x\sqrt{-g}f(R)$	$\left\{\begin{array}{ll}\mathcal{L}_m(g_{ab},\psi) & \text{metric}\\ \mathcal{L}_m(g_{ab},\psi) & \text{Palatini}\\ \mathcal{L}_m(g_{ab},\Gamma^a{}_{bc},\psi) & \text{metric-affine}\end{array}\right.$
DGP	$\frac{1}{2\kappa_{(5)}}\int d^4x\sqrt{-\tilde{g}}\tilde{R}+\frac{1}{2}\int d^4x\sqrt{-g}R$	$\mathcal{L}_m^{brane}(g_{ab},\psi)$
GB	$\int d^Dx\sqrt{-g}\left(R^2-4R^{ab}R_{ab}+R^{abcd}R_{abcd}\right)$	$\mathcal{L}_m(g_{ab},\psi)$
Galileons	$\frac{1}{2}\int d^4x\sqrt{-g}\left(R+\sum_i^5 c_i\mathcal{L}_i\right)$	$\mathcal{L}_m(f(\pi)g_{ab},\psi_i)$
HL	$\frac{1}{2}\int dtd^Dx\sqrt{-g}\left((\dot{\Phi})^2-\frac{1}{4}\Phi(\nabla^2)^2\Phi\right)$	$\mathcal{L}_m(g_{ij},N,N_i,\psi)$
RS	$\frac{1}{2\kappa_{(5)}}\int d^5x\sqrt{-g_{(5)}}\left(R-2\Lambda_{(5)}\right)-\int d^4x\sqrt{-g}\sigma$	
KK	$\frac{1}{2\kappa_D}\int d^Dx\sqrt{-g_D}R_D$	$\mathcal{L}_m(g_{ab},\psi)$

where ψ is a matter field,Φ is an exotic scalar field, ω is the Dicke coupling constant, A^a is a spacetime 4-vector field, $N^i(x,t)$ is a shift vector, $N(t)$ is a homogeneous lapse function, π is a galleon field, σ is brane tension. $\mathcal{L}_g$, $\mathcal{L}_s$ and $\mathcal{L}_g$ are the actions for the metric (tensor), scalar and vector fields, respectively. κ_D is the bare gravitational constant in D-dimensions. $\tilde{g}$ and $\tilde{R}$ represent the determinant of the spacetime metric in D-dimensions and its corresponding Ricci scalar.