Ancient Mesopotamian’s system of measurement: possible applications in mathematics and physics teaching

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Abstract. The study of the ancient Mesopotamian units of measurement and mathematics provide several components that could be used for the teaching of physics, mathematics and history of science. By studying the known clay tablets, it is possible to learn their mathematics, how the art of teaching began, how the first numbers were represented and what were their purposes, as the ancient calculus of area, volume, weight, time and others and to learn the origin of other mathematical and scientific concepts. The circle with 360 degrees, watches with 12 divisions and the year made of 12 months are some examples of the influence of the Mesopotamian’s mathematics and system of measurement in our culture, this can be very interesting for students who have no interest in math and physics classes and even more interesting for students who already have interest in these areas. The purpose of this paper is to show some of the ancient Mesopotamian units of measurement and mathematics with their possible uses in education.

1. Introduction

Many students have no interest in physics classes, either because the subject is scientific and involves mathematics or because of the traditional methodology used to teach it. Another factor that leads students to become uninterested in physics is that they generally do not learn the origin of the knowledge to be learned or what are their applications; this makes the approach to teaching physics with the history of science very interesting, especially when it comes to more fundamental and arbitrary topics, such as in the case of units of measurement. This leads us to the creation of this article, which has the objective to provide knowledge that teachers can bring to their classes and make them more attractive to their students.

2. Historical Introduction

2.1. The Mesopotamian civilizations

According to [1], the first written documents found in Mesopotamia date from approximately 3500 BC and these writings were found in recent centuries and revealed many things previously hidden and unknown. The form of writing engraved on clay tablets was called cuneiform because the traces of the characters are similar to nails or wedges (lat. Cuneus). The first nation known to use this form of writing were the Sumerians, a small nation of unknown origin who lived on the shores of the Persian Gulf, along the lower course of the Euphrates. Their culture peaked around 2000 BC, after reclaiming their lands that had been conquered by the Akkadians. The Sumerians disappeared in the 19th century BC, with the beginning of the Amorites dynasty, who were concentrated in Babylon (making the people also known as Babylonians) and had as main representative King Hammurabi, such people remained until the 6th century BC until being completely defeated by the Persians. These civilizations were in constant wars and always preserving the culture of the Sumerians and the additions made by other civilizations, which makes it difficult to assign the achievements of each of people in the Mesopotamian culture. To facilitate the understanding of the main civilizations and their respective approximate periods in Mesopotamia, table 1 summarizes the data provided by the tables from [2].

Table 1. The main civilizations that inhabited Mesopotamia in Ancient Times and their respective periods of permanence in the region.

Civilization	Period (Estimated)	Name of the Period
Sumerians	4000 – 2334 BC then 2100 – 2000 BC	
Akkadians	2350 – 2150 BC	
Assyrians	2500 – 1595 BC then 1100 – 539 BC	Neo-Babylonian Empire (625-539 BC)
Persians	539 – 333 BC	
Amorites	1894 – 1595 BC	First Babylon dynasty
Kassites	1595 – 1155 BC	
Seleucid Empire	333 – 150 BC	Seleucid Empire in Babylonia

According to [2], even with the continuous conflicts among those civilizations "(…) there remained in the area a sufficiently high degree of cultural unity to justify referring to the civilization simply as Mesopotamian".

2.2. History of the writing

As previously stated, the first written documents date from 3500 BC, and are engraved on clay tablets, which, still according to [2] were carved with wedge-shaped objects in heated clay and then left in the sun or in ovens to dry. This made these documents much less vulnerable to the ravages of time than the Egyptian papyri, which allowed for the discovery and acquisition of many more written records from Mesopotamia than from Egypt, despite being older, what makes many researchers believe that the cuneiform writing can be the oldest form of written language.

It is also believed that the first written registries were made for accounting purposes, either as a receipt for transactions (mainly involving livestock), or for the control of distribution of commodities or food for slaves.

Figure 1. Agricultural account tablet in Sumerian referring to flocks and herds. From Southern Babylonia. About 2400 BC. Harry Ransom Center: Gutenberg Bible Primary Source Education Module http://www.hrc.utexas.edu/exhibitions/education/modules

Due to the early writings had a function of accounting, the numeric symbols appeared along with the phonetic symbols, or even earlier, considering the pictographic writing period [3], where there were no letters, but numbers and drawings were made for memorization and counting purposes. It is also believed that shortly after the invention of writing, the Sumerians developed the so-called "Science of lists" [1], which basically attempts to annotate and list knowledge in a categorized way. Such lists were used not only for writing but also for sciences and knowledge in general, as well as practical consultations and education, most of these clay tablets had seals of authenticity from the scribe who wrote them.

One good example of this are the multiplication tablets that consists of lists of some multiplications of numbers, just like the multiplications tables that children learn and practice at school nowadays. That is the origin of the term "multiplication table". Another good example of this type of list to be cited is a tablet compiling lexical equivalents between the Sumerian and Akkadian languages, which shows the adoption of the Sumerian culture by the Akkadians after their domination of the region.

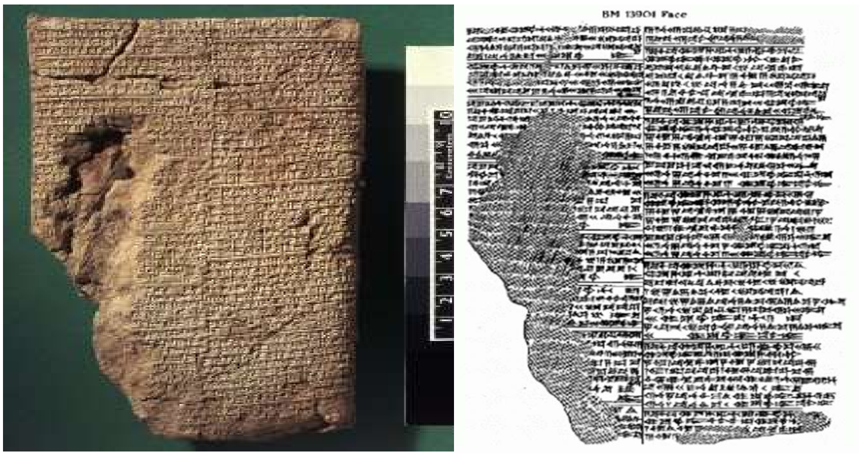
3. Mathematics

In mathematics, two types of texts are known: the numerical tablets and the problem tablets. The numerical tablets which are hardly different from the modern tables, have numbers arranged in ordered columns in increasing or decreasing series, references from one table to another, combinations, and so on. The problem tablets are basically collections of exercises, similar to those of textbooks, which had precisely this purpose but which had in many cases lacked information and specifications that should be indicated orally to the student, what shows the absence of indeterminacy between written and oral language, and also the fact that mathematical language was still incomplete. One good example of this would be the absence of arithmetic symbols. Many of the tablets have a wide range of exercises, usually of the same kind and that must be solved by identical methods, changing only the coefficients of the accounts. The exercises were numbered and clearly separated by single or double lines.

The problem tablets reveal the daily life of the population. In them, we find problems that contained the most diverse subjects: areas of land, grain yield produced in a field, constructions of channels and dykes, measures of solid cylinders, loans, etc. The operation to be performed came with the mathematical context of the problem to be solved or was explicitly written in the tablet or in the problem, which was mainly due to the absence of arithmetical symbols in the mathematical language of the time. One good example is the exercise 7 of the BM 13,901 tablet (Figure 2), extracted from [1]:

"I added seven times the side of my square, and eleven times its surface. This resulted in 6.15 (6.25 in decimal notation)."

According to [1], in this exercise the goal is to obtain the value of the side $\mathbf{x}$ of the square, in the equation $\mathbf{11x^2 + 7x = 6.15}$, which must be shaped and answered by the student. In this exercise, it is possible to perceive the dependence of the oral explanation and contextualization of the goal, being written only the fundamental elements of what is desired, but without sufficient connections for the understanding only from reading.

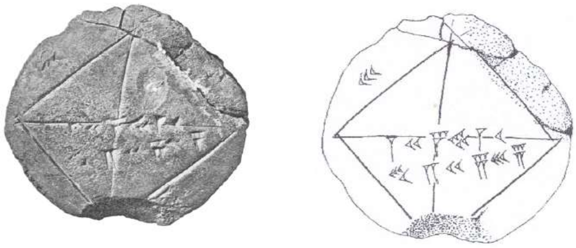


a) b)

Figure 2. The BM 13,901 tablet, from the British Museum (a) [1896,0402.1, AN324556001], and its representation (b), where it is possible to see the double lines separating the exercises.

3.1. Geometry.

In Mesopotamia the mathematical tradition is almost purely algebraic, so there was not much "geometrization" of the problems in the exercises and mathematical problems. The accounts were made by algebraic relations that is unknown how they were obtained. For example, they discovered the relation of the Pythagorean theorem, but not studying triangles, but rather wanting to draw a diagonal to rectangles.



a) b)

Figure 3. a) A clay tablet picture from ancient Babylon extracted from [4], that can be found in Yale University Collection (Nº. 7289), b) and its representation from [5], that contains the calculation of $\sqrt{2}$, using the diagonal of a square, with up to three sexagesimal digits, deferring for about 0.000008 of the currently known value, according to [6].

Other geometric relations were also known, but the solution to the exercises was algebraic, rarely having illustrations without regular geometry and which was not useful in resolution, according to [1]. One example of this can be imagined as the case of the drawing of a stone to have its mass calculated, where the drawing have only illustration propose.

3.2. Arithmetic.

The Babylonian numeration presents two original characteristics, which are not found in any of the ancient systems: Position numbering and with sexagesimal base. The position notation is opposed to the principle of juxtaposition, which served as the basis for all other ancient systems, as the Indo-Arabic system.

In the juxtaposition system, the value of a number corresponds to its value multiplied by a power value of its base arbitrated by its position, so all the numeric slots included in the value must be filled, even if with 0, to represent that a certain place is null, for example, in the number 607, there are 6 hundreds, 0 tens and 7 units, if the 0 were not present, the number would be 67, and understood as 6 dozens and 7 units, what corresponds to another value, what does not happen in the position numbering system.

In the position system, the principle is practically the same, but there is no obligation to occupy all the numerical positions, such as there was still no symbol for zero, so there is no certainty in the value of a number only from reading. Then, following the previous example of number 607, and using our decimal system, but in a position system, this number would be written as 67, as well as the numbers 6007, 6070, 60007 and a multitude of other numbers in this format, making the interpretation of a number depend on its context or oral confirmation.

The difference between the Mesopotamian sexagesimal base and the Indo-Arabic decimal base is the value of the base that multiply the numbers from the second slot from right to left, without dots. Then when writing 11 in the decimal system, we have 1 unit and $1x10^1$, which corresponds to a ten plus one, that results in eleven units. On the sexagesimal basis, 11, can mean the number eleven or 1,1 which corresponds to 1 unit and $1x60^1$, so 61 units, considering that there are no null slots between the ones. Therefore, in the second slot to the left of the unit, the decimal system multiplies that number by 10^1, while in the Mesopotamian sexagesimal system this number is multiplied by 60^1. When adding a third slot to the left, as in 111, which is $1x10^2 + 1x10^1 + 1x10^0 = 100 + 10 + 1 = 111$, differently from the sexagesimal base, where 1,1,1 represents $1x60^2 + 1x60^1 + 1x60^0 = 3600 + 60 + 1 = 3661$ (or 1,11 and 11,1) if there are no null slots between the ones. Sometimes the null slots were represented by a larger spacing between the numbers before the creation of a numeral for the zero (that will be explained below).

According to [8], although the Sumerian numeration is sexagesimal and position, the Babylonians used other counting systems (at least five), but in mathematical and astronomical texts the notation is strictly of position and sexagesimal. The counting system used depended on what was counted, among the other systems besides the sexagesimal, it is important emphasizing the bisexagesimal, that was used to count grains, cheeses and fresh fish, for example. While animals, humans, animal products, dried fish, fruits, tools, stones,vessels and others were quantified using the sexagesimal system.

One possible reason for choosing a sexagesimal system may have been, in addition to the large number of submultiples of number 60, the possibility of a system where it is possible to divide grains for both the 30-day months and 12-months year easily.

4. Numbering System.

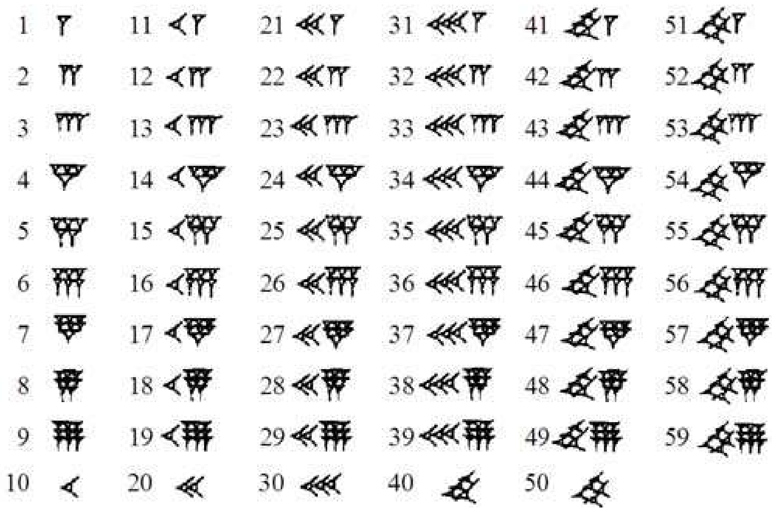


Figure 4. Babylonian numerals, available in https://commons.wikimedia.org/wiki/File:Babylonian_numerals.PNG en:sugarfish [Public domain], via Wikimedia Commons.

According to [1], the Babylonian notation has two gaps, on the one hand, inside each order of units, the notation is decimal and juxtaposition, the unit and the ten being the only symbols used, and a symbol for zero was found only in astronomical texts, in Seleucid times. Approaching to that subject, after displaying which symbols were used for zero, [10] has made the following quotes:

> "(...) In the Seleucid period, instead of white space, a symbol appears (...) which in other contexts was only a separation mark in cuneiform writing but which is also never used at the end of the number representation." (Quoted from [8])
>
> "In the III century BC a character of the shape of two linked beams, was used to mark a missing space in the middle of numbers; this character was not used to the right of a number to require an order of magnitude; the notation remained so ambiguous." (Quoted from [9])

a) b)

Figure 5. a) anb b) are the representations of zero according to [10].

Even in the low period where the zero received a symbol, it was never used in the end of a number. The Babylonian astronomers wrote 0,25 but never 25,0. It is necessary to avoid the conclusion of the functional identity of our zero to the Babylonian zero. Before its invention, people felt the necessity to indicate the orders that were missing, in some tablets there are spaces in the places where the symbol "0" would be located.

The absence of zero in the old texts renders ambiguous the absolute value of the unit, which can mean both 1 and 1,0 ($1x60^1 = 60$), among an infinite number of possibilities, what constitutes a

tiresome obstacle to the deciphering of mathematical texts. This obstacle can be easily comprehended with the example of the number 1,20,3 (wrote in figure 6) which can be read as:
1,20,2,1, i.e:, $1x60^3 + 20x60^2 + 2x60^1 + 1x60^0 = 216000 + 72000 + 120 + 1 = 288121$, or as 1,22,1, i.e: $1x60^2 + 22x60^1 + 1x60^0 = 3600 + 1320 + 1 = 4921$, or even as
1,23 = $1x60^1 + 23x60^0 = 60 + 23 = 83$, among an infinity of fractional possibilities or with null slots in some order of magnitude due to absence of zeros.

Can be read as:

1,20,3	1,20,1,1,1
1,10,10,3	1,20,2,1
1,10,13	1,20,1,2
1,10,11,2	1,21,2
1,10,12,1	1,22,1
1,10,10,1,1,1	1,23

...

Figure 6. Cuneiform number from the example and some possible interpretations of it. For better understanding, check the numbers 1, 2, 3, 10, 11,12,13,20, 21, 22 and 23 in figure 4.

Because it is a sexagesimal system, the number 0,15 represents $\frac{15}{60}$ and not $\frac{15}{100}$ as in our decimal system. To represent fractions of higher order was used the representation 0.0,15, which is equivalent to $\frac{15}{60^2} = \frac{15}{3600}$. The divisions of the unit by 2,3,4,5 and 6 results respectively in 0,30, 0,20, 0,15, 0,12 and 0,10. It is unknown exactly what the Mesopotamian knowledge in operations involving fractions was. And it's also important to emphasize that the fractional notation with zeros began only in the Seleucid times with the numeral for zero, before that times, the context was needed to know if a number was whole or fractional, what means that 0,15 was written exactly the same way as 15.

5. System of measurement.

The ancient Mesopotamian system of measurement is made of units with respect to multiples and submultiples (with sexagesimal relations) with a main and standard unit of measure, the cubit (that is the distance between the elbow and the tip of the middle finger), which had several different values over time, being generally arbitrated based on measures of the king's or princes' cubits, causing the value of all other units to consequently change.

According to [1], there are three fundamental units: cubit for length, ca for volume and pound for weights. The ca and the pound are submultiples of the cubit, the ca being the value of the 144^{th} part of the cubic cubit and the pound being the weight of a volume equivalent to the 240^{th} part of the same cubic cubit.

5.1. Obtaining the value of the fundamental units.

The study of a graduated ruler found on the lap of a statue of a Lagash prince, Gudea (Figure 7) and the data collected by the archaeologist also measuring the base of the current "Tower of Babel" whose dimensions in ancient measurements are provided in a cuneiform text, reveal that the cubit was about 50cm.

Figure 7. Sitting diorite statue of Gudea, prince of Lagash, dedicated to the god Ningishzida, c. 2120 BC (neo-Sumerian period). Excavated in Telloh (ancient Girsu), Iraq. On display at the Louve, Department of Oriental antiquities, Richelieu, ground floor, room 1. Public Domain.

Among other findings, a vessel with an indication of its capacity in the bottleneck, allowed to evaluate the ca as being worth more or less 8.40dl. To find the value of the pound, weights (that have their values inscribed and were found in ruins) were placed in scales to find an approximate value of 505g.

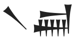
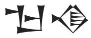
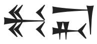
Figure 8. Mesopotamian duck weights made of Hematite from the 2nd millennium BC, displayed in the The Oriental Institute of The University of Chicago (OIM A9684).

Figure 9. Babylonian black stone duck weight with 2 cuneiform written lines from the British Museum (1963,0715.1, AN949886003).

5.2. Some of the known units of measurement.

Table 2. Table containing some units of known Mesopotamian measures, such as their approximate values and their cuneiform writing, according to data extracted from [1; 11].

Unit	Ratio	Ideal Value	Cuneiform	Measure
Finger	1/30	0.015m		
Foot	2/3	0.333m		
Cubit	1	0.497m		
Step	2	1m		
Reed	6	3m		Length
Rod	12	6m		
Cord	120	60m		
Cable	720	360m		
League	21600	10,800m		

Garden	12x12	36m²		
Quarter-field	60x60	900m²		
Half-field	120x60	1,800m²		Area
Field	120x120	3,600m²		
Estate	360x720	64,800m²		
Ca	1	0.084L	Not found	
Bowl	12	1L		Volume
Vessel	120	10L		
Bushel	720	60L		
Grain	1/10800	46.6mg		
Shekel	1/60	8.40g		Mass
Pound	1	500g		
Load	60	30kg		
Gesh	1/360	240s		
Watch	1/12	7,200s		
Day	1	86,400s		Time
Month	30	2,592,000s		
Year	360	31,104,000s		

6. Inheritances of the Mesopotamian civilization.

Besides the invention of the wheel, of the arithmetic operations, of the writing and the rivers transposition technique, according to [12], quoted by [10], from the Mesopotamian civilization we received a considerable number of our most common cultural elements: the year of twelve months and the week of seven days; the fact that the dials of our watches contain the numbers of one to twelve, corresponding to the Chaldean division of the day in twelve double hours; the belief in horoscopes; the superstition of planting according to the phases of the moon; the twelve signs of the Zodiac; the 360 degree circle.

7. Conclusions

Mesopotamian civilizations have created many of the bases of mathematics, the calendar, the way of seeing the world, the formation of the first cities, the writing, and many other things. Such creations deserve greater recognition and the subjects presented certainly increase the students' interest, thus making it important that at least teachers have knowledge to be able to explore directly or indirectly in the classroom. The mathematical part, and their system of measurement as a precursor of the International System of Units, such as from didactic exercises of the time that can also be used, outside the cultural elements that can be also passed on to the students. In addition, the (even superficially) study of mathematics evolution and systems of numeration, measurement and scientific development can be very rich enabling teachers to show students that science is constantly evolving, with no permanent truths or definitive systems.

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