Buquoy's problem in an introductory physics course

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Abstract. The Buquoy's problem represents an interesting example of a one-dimensional motion in which a uniform thin fibre is pulled upwards from a horizontal plane by a constant vertical force exerted against the homogeneous gravitational field. The problem includes several interesting concepts that are developed within the introductory university physics course, like motion with a variable mass, the use of the effective potential and a solution of a non-linear ODE of the second order, which provides an opportunity for some numerical modelling of the motion. It is a specific example of damped oscillations and according to our experience it can be discussed e.g. within a course in classical mechanics. Our contribution concentrates on the correspondence between a theoretical model and the experimental realization of this problem, which can be demonstrated in a standard classroom. In our set, the upward force is represented by buoyancy acting on a helium filled balloon. The experiment helps to develop some other competences of the students and naturally leads to a little bit more general mathematical model with an equation of motion that includes the constant mass of the helium ball and the dumping resistance force.

1. Introduction: the Buquoy's problem history and formulation

Mechanical problems dealing with variable masses represent an integral part of university courses in classical or theoretical mechanics. Typically, they are related to the rocket motion (see e.g. [1–3]), some textbooks provide several other examples (see e. g. problems 26.8 "condensation of a water droplet" or 26.9 "motion of a truck with variable load" in [4]).

In our contribution we would like to discuss another problem from the variable mass category with rather long history and several interesting details that might be of pedagogical interest for both our colleagues teaching undergraduate courses, and for their students. The topic is associated with a Czech aristocrat, mathematician and a talented inventor, a member of the French aristocratic family Buquoy de Longueval, Jiří František August Buquoy (Graf Georg Franz August von Buquoy in German). Born in 1781 in Brussels, he received his education in Vienna and Prague and became interested in natural sciences, mathematics, philosophy, economy, legal issues, literature, and poetry. With the wealth he inherited from his uncle, J. F. A. Buquoy was able to publish his own works, so since 1803 he was taking care of the large family possessions and his own investigations. Between 1810 and 1818, he was involved with the theory and practice of steam machines and in 1810 he introduced an original steam engine providing the output of 17 HP (i.e., about 12.7 kW). Following the traditions in the area of his estates, he was also involved in glass production and in 1817 he invented an original process technology

(now forgotten) of a black opaque glass called hyalite. Moreover, in 1838 he founded the oldest nature reserve in continental Europe (Žofín Forest in South Bohemia).

Between the years 1811 and 1815, J. F. A. Buquoy was involved mostly in issues of applied mechanics. He was the first one to formulate the problems and the equations of motion connected with the dynamics of variable mass systems. In 1812 and 1814 he issued two books on this subject and in August 1815 he presented his findings at the Paris Academy of Sciences. One of Buquoy's explicit examples of a system with varying mass described in 1814 [5] is as follows:

Consider an ideally flexible fibre lying reeled on a horizontal plane. Determine its motion when a constant vertical force (directed upward) is exerted on the end of the fibre.

Although the solution of the problem suggested in [5] was not correct, the idea of such motion remains connected with Buquoy's distinguished personality and since that time the problem has been dealt with many more times. In this text we build especially on the systematic analysis done by Šíma and Podolský [6]. Starting with the situation depicted in Figure 1, it is natural to introduce the following simplifying assumptions:

- the vertical gravity field is homogeneous;
- the fibre is thin and its linear density η is constant;
- the fibre reels off without friction at the origin of the *y*-axis during the upward motion;
- the fibre "smoothly disappears" at the origin of the *y*-axis during the motion downward, i.e., the part that has already landed on the plane does not move.
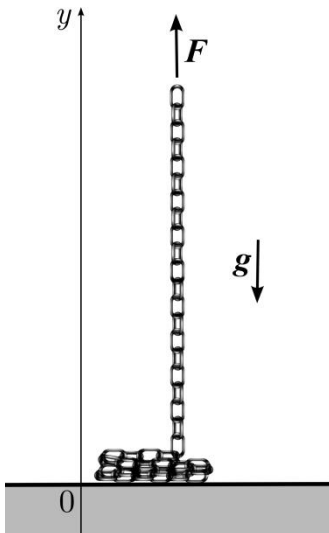

Figure 1. A schematic picture illustrating the formulation of the Buquoy's problem with a vertically moving fibre (or a chain) under the constant force F in a homogeneous gravity field with the gravity acceleration g. The actual position of the upper end above the horizontal plane is denoted by *y*.

The paper is organized in the following way: in section 2 we resume the equation of motion for the Buquoy's problem, the main characteristic of the solution and the concept of the effective potential which can be employed for the description of motion. In section 3 we describe the experimental realization of the motion and the values of basic materials and setup constant parameters. In section 4 we discuss a fit of video analysis data to a mathematical model which leads to a more general equation of motion. In the conclusion we try to formulate the main findings based on the experiment and experience with how the problem is integrated into our undergraduate curriculum.

2. Equations of motion, the effective potential and modelling of the motion

The Buquoy's problem provides several interesting features that give a good reason to use the problem within the undergraduate physics course. As shown e.g. in [6], the differential equations are different for upward and downward motion; for downward motion the appropriate momentum absorbed by the
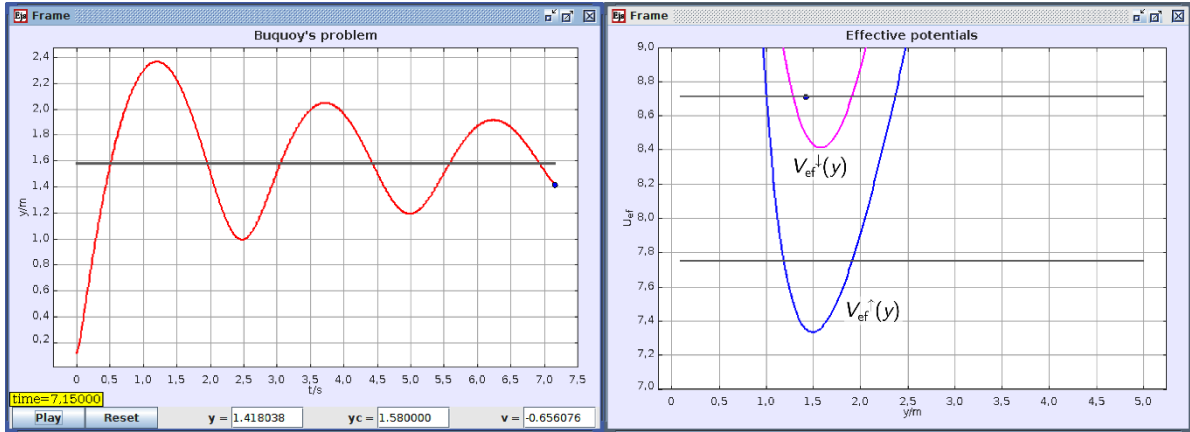
Figure 2. A simulation of a simplified model with *Easy Java Simulations* (http://fem.um.es/Ejs/) used for our seminars in classical mechanics ($y_0 = 0.12$ m, $\dot{y}_0 = 0$).

"infinitely massive" horizontal plane. Denoting the linear density of the fibre as η, the moving vertical part of the fibre of the length y has the mass ηy. Within the aforementioned simplifying assumptions, the equation of motion

$$\frac{\mathrm{d}(\eta y \dot{y})}{\mathrm{d}t} = F - \eta y g, \tag{1}$$

can be transformed into a compact form [6]

$$\ddot{y} = g\left(\frac{y_c}{y} - 1\right) - \frac{1}{2}(1 + \mathrm{sgn}\,\dot{y})\frac{\dot{y}^2}{y}. \tag{2}$$

This equation, which includes an interesting dependence on the velocity, has a stationary solution $\dot{y} = 0$, $\ddot{y} = 0$ for a set of initial conditions $y_0 = y_c$, $\dot{y}_0 = 0$, with the value of y_c determined by the force F and the gravity acceleration $g \approx 9.81$ m s^{-2} (for the Czech Republic). This stationary solution

$$y_c = \frac{F}{\eta g}, \tag{3}$$

corresponds to an equilibrium situation where the force F is equal to the gravity force acting on the vertical part of the fibre with the length y_c.

The solution of equation (2) is generally not possible to express in an analytical way, but it is possible to get a numerical solution with the help of suitable software. Generally, the numerical modelling of physical phenomena is an important part of undergraduate courses. As it has been stressed in various contexts, mathematical modelling of the physical world should be the central theme of physics instruction (see e. g. [7]). Visualization of equations (the ODE in this case) is very important to make more abstract concepts (like the effective potential here) visible. In our courses (both on classical mechanics or within a propaedeutic course on the mathematical background of physics) we prefer the approach to let our students themselves create their own simulations in the software environment they prefer and provide them with the corresponding support. This approach has the advantage of getting the student to work in an exploratory and constructive way. In literature we can found various approaches and techniques to solve ODEs in physical problems, from Excel sheets (e. g. [8−9]) to Coach (e. g. [10]), MATLAB or GNU Octave (e. g. [11−12]), Python (e. g. [13]) or Easy Java Simulations (e. g. [14−16]). Of course, there are also complex computing systems like Wolfram Mathematica, which we also use in our courses. Not surprisingly, within physics courses we prefer any system that does not require an enormous investment of time to create a graphical simulation and enables to concentrate on the underlying physical model (an example of a model for the Buquoy problem based on the equation (2) is in Figure 2, it has been developed within our seminar on classical mechanics).

Though we can visualize the ODE's solutions numericaly, the overall character of the motion can be found with the help of the effective potential concept, which in our classical mechanics course is

introduced in connection with the Kepler's problem and the planetary motion in the central force field (see, e.g. [4]). Both for upward and downward motion the effective potential has a form of a potential well (see Figure 2) with the analytical forms [6]

$$\frac{1}{2}\dot{y}^2 + V_{\text{ef}}^{\uparrow}(y) = \frac{F}{2\eta}, \qquad V_{\text{ef}}^{\uparrow}(y) = \frac{g}{3}y - \frac{C}{y^2}$$

for the upward phase, where the constant C is determined by the initial conditions; in fact, for the form of a well-like potential we must have $C < 0$. Having a constant mechanical energy for a run of the upward motion, we can see (Figure 2) that the motion comes to the turning point $\dot{y} = 0$ from which the downward motion follows. The effective potential of the downward phase reads as [6]

$$\frac{1}{2}\dot{y}^2 + V_{\text{ef}}^{\downarrow}(y) = V_{\text{ef}}^{\downarrow}(y_{\text{max}}), \qquad V_{\text{ef}}^{\downarrow}(y) = gy - \frac{F}{\eta}\ln y,$$

where y_{max} is a coordinate of the top turning point (in which the preceding upward motion ends). The effective potential has the same value $V_{\text{ef}}^{\downarrow}(y_{\text{max}})$ in the bottom turning point $\dot{y} = 0$ as well, in which the motion switches into the upward phase again.

Thus, we get an oscillating motion around the equilibrium height y_{c}, which is similar to the damped harmonic oscillator. The decreasing amplitude is caused by the loss of momentum absorbed by the ground plane during the downward motion. The concept of the effective potential enables us to find the maximal height y_{m} of the first top turning point from the condition

$$V_{\text{ef}}^{\uparrow}(y_{\text{m}}) = \frac{g}{3}y_{\text{m}} - \frac{C}{y_{\text{m}}^2} = \frac{F}{2\eta} = \frac{g}{2}y_{\text{c}},$$

that leads to a quadratic equation with generally two solutions corresponding to the top and the bottom turning points. For the initial conditions $y_0 < y_{\text{c}}$ relevant to the experiment set described in section 3, we obtain the maximum height

$$y_{\text{m}} = \frac{1}{2}\left(\frac{3}{2}y_{\text{c}} - y_0\right) + \frac{1}{2}\sqrt{\frac{3}{2}(y_{\text{c}} + y_0)^2 - 4y_0^2}. \tag{4}$$

Let us point out that for the initial condition $y_0 > y_{\text{c}}$ the maximum height is given by y_0.

The motion has one more interesting feature that can be easily derived from the equation of motion (2). For small oscillations around the equilibrium value y_{c} we can introduce a new variable $\xi = y - y_{\text{c}}$ ($\xi \ll y_{\text{c}}$) and by neglecting the small term in the equation (2), we arrive to a linearized form

$$\ddot{\xi} + \frac{g}{y_{\text{c}}}\xi = 0.$$

Therefore, the solution of the Buquoy's problem has a harmonic-like character in the small oscillations limit with a period of small oscillations

$$T_{\text{so}} = 2\pi\sqrt{\frac{y_{\text{c}}}{g}} = \frac{2\pi}{g}\sqrt{\frac{F}{\eta}}. \tag{5}$$

This limit might be important for a discussion in the class, as it is another supporting way to demonstrate the oscillatory character of the motion.

One of the required features of the physics education is a strong link between theoretical models and a real practical measurement. The analysis of the Buquoy's problem in [6] includes a suggestion of a possible experimental set: "*The problem could be modelled experimentally as the vertical motion of a balloon with a heavy rope hanging down. The constant upward force would be the buoyant force exerted on the balloon in air. The mass of the balloon and the friction forces would have to be neglected.*" As

Figure 3. The experiment setup in an equilibrium position.

shown in the following section, the assumption that we can neglect the mass of the balloon and the dumping forces is not so obvious and straightforward.

3. The experiment

Our goal was to construct a functional Buquoy's oscillator for the classroom demonstration and to compare its behaviour with the mathematical model described above. When designing an experimental model, first it was necessary to determine or estimate some important parameters of the experimental set and its components.

A rubber balloon filled with helium was chosen as the source of a constant tensile force F. To simplify the supporting calculations, we assumed that that balloon had approximately a spherical shape. Because our intension was to show the experiment in a standard classroom with a maximum ceiling height of about 330 cm, for the approximation of the linear density we took the value $y_c = 2$ m; than the maximum height can be estimated by putting $y_0 = 0$ m in equation (4), after which we obtain $y_m \approx 2.72$ m.

The average diameter of the balloon was determined from its circumference as $d = 30(2)$ cm, i.e. its radius value was $r = 15(1)$ cm. Under the simplifying assumption of the spherical shape, the volume of the balloon was $V \approx 14(3) \times 10^{-3}$ m^3. According to Archimedes's principle, the buoyant force exerted on the helium filled balloon with the volume V in the air with the density $\varrho = 1.2$ kg m^{-3} is given by the famous relation $F_B = V\varrho g$; for the Buquoy's problem this force simultaneously plays the role of the tensile force F acting on the balloon and the fibre. Substituting into the equation (3), the linear density can then be expressed as

$$\eta = \frac{F_B}{g y_c} = \frac{V\varrho}{y_c}.$$

For our classroom-size installation, we could estimate the linear density of the fibre to be $\eta = 9.11$ g·m^{-1}.

To meet the assumption of a homogeneous fibre with a constant linear density, it should ideally be made from small interconnected parts which can smoothly depart from or fall on the ground plane. Finally, after some experimental attempts, a chain made of small balls, which is usually used in a bathroom sink, was chosen for this purpose. Particularly, we selected the chain with the smallest balls, the linear density of which was determined by digital scales to be $\eta = 10.40(1)$ g m^{-1}.
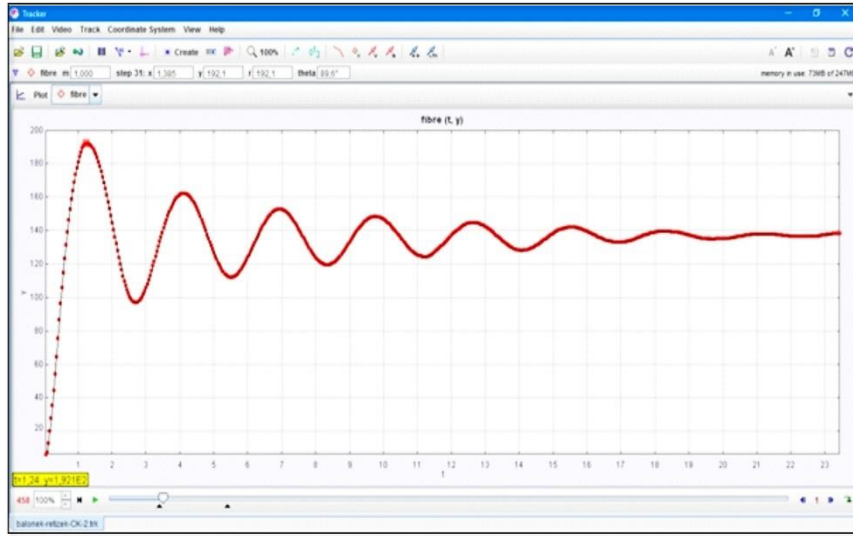
Figure 4. Video data processing with Tracker (https://physlets.org/tracker/), a free video analysis and modelling tool.

Our Buquoy's oscillator in the equilibrium position $y = y_c$ is shown in Figure 3. In all of our seven measurements, the balloon was released at a position $y < y_c$ with zero initial velocity and its motion was recorded by a video camera. Performing a video analysis of the motion with a free video analysis and modelling tool *Tracker*, we got a resulting graph in Figure 4, which shows the time dependence of the y coordinate marking the end of the fibre (chain). From the graph we can estimate the stationary position $y_c = 1.38$ m. Measuring the net tensile upward force by Vernier Dual-Range Force Sensor $F = 0.136(6)$ N and using the linear density $\eta = 10.40$ g m^{-1}, mentioned above, we can calculate the value y_c substituting into the equation (3), which gives the result $y_c \approx 1.33$ m, which is by 4 % less than the value read from video analysis data.

The graph in Figure 4 also enables us to guess the height reached by the chain upper end for chosen initial conditions as $y_m = 1.92$ m. If we assume that $y_0 = 0$ m, we can rewrite the equation (4) in the form

$$y_m = \frac{3}{4}\left(1 + \sqrt{\frac{2}{3}}\right)y_c \approx 1.36\, y_c.$$

For the graph-deduced value $y_c = 1.38$ m we get $y_m = 1.88$ m, which is approximately 2 % less than the experimentally determined one. So, the experiment demonstrates that the relation between values of the y_c and y_m parameters read from video analysis quite corresponds to the relations of these values derived from the theoretical model.

Figure 5 gives a comparison of two plots – one represents the theoretical model described by the equation (2) and the other represents the data extracted from the video record. At first glance one can see that there is a significant difference between them. Though the overall character of the motion corresponds to the dampened quasi-periodic oscillations predicted by the mathematical model described in section 2, the amplitudes and periods do not match, which means that our experiment needs a slightly different, less simplified mathematical model introduced in section 4.

We can also estimate the period of small oscillations following the equation (5). Substituting the values $F = 0.136$ N and $\eta = 10.40$ g m^{-1} into equation (5), we get $T_{so} = 2.31$ s. The average value of the period in Figure 5 is $T_{exp} = 2.85(2)$ s. The difference of 19 % gives us further motivation to search for a more satisfying mathematical model describing our measurement. Certainly, part of the difference can
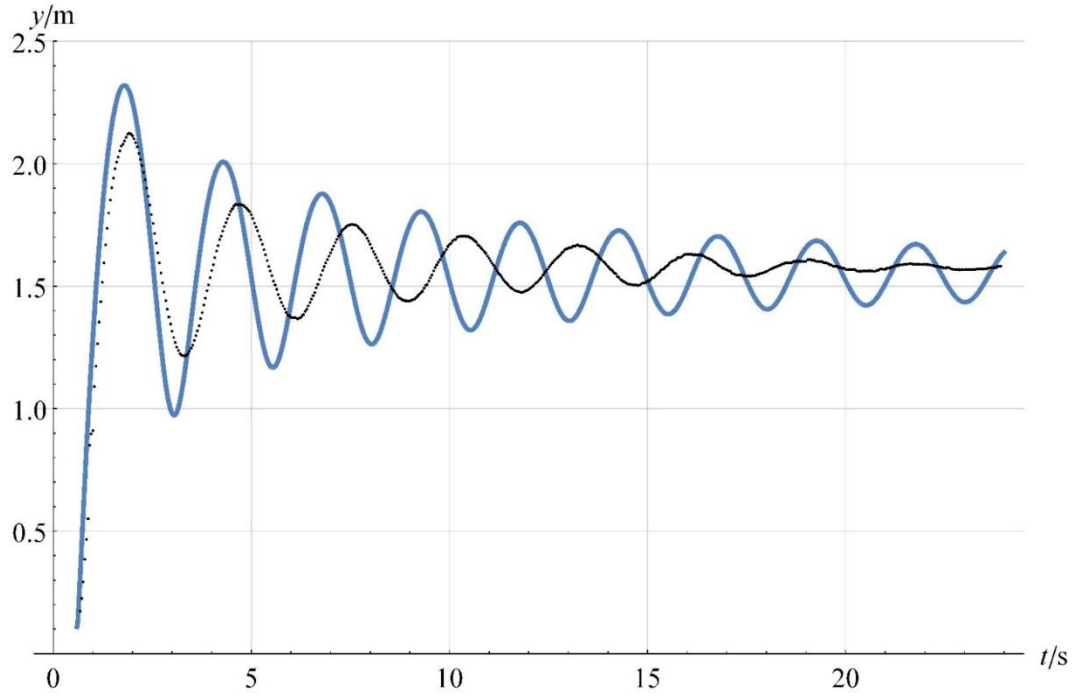
Figure 5. Plots of the simplified theoretical model numerical solution (blue curve) and the video analysis data (black dots) for the same initial conditions; the difference signalizes that the simplified model is not suitable for the full description of our experiment.

be explained by the fact that the movement of the balloon and the fibre was not absolutely horizontal and perpendicular to the surface of the table; in some moments the balloon also moved horizontally, so the fibre was pulled over the pad.

4. Fitting the measured data and a generalized equation of motion

To find a better match between the theoretical model and experimentally determined values, we need to consider two other factors that were not included in a simplified model in section 2. Firstly, the air resistance and the fibre friction slow down the motion of the balloon. Secondly, the mass of the balloon m_b cannot be neglected in comparison to the mass of the vertical part of the fibre. Therefore, we must include these two extra factors in the theoretical model. Analogically to a damped harmonic oscillator problem (see, e.g. [1−2]), the resistance damping can be represented by a force depending on the fibre velocity linearly. Then the equation of motion (1) can be modified as

$$\frac{\mathrm{d}(\eta y \dot{y} + m_b \dot{y})}{\mathrm{d}t} = F - \eta y g - \gamma \dot{y},$$

where γ is the coefficient of the dumping force dependence on velocity. Straightforwardly, the generalized form of the equation (2) has the form

$$\ddot{y} = \frac{1}{y + \alpha}\left[g(y_c - y) - \frac{1}{2}(1 + \mathrm{sgn}\,\dot{y})\dot{y}^2 - \beta\dot{y}\right], \tag{6}$$

where $\alpha = \frac{m_b}{\eta}$ is the factor related to the mass of the balloon pulling the fibre, $\beta = \frac{\gamma}{\eta}$ is the coefficient related to the dumping force.

For further proceedings, Wolfram Mathematica computing system (version number 11.2.0.0) was employed to find the best least squares fit for the introduced parameters α and β. Its procedure *NonlinearModelFit* enables to fit data to a model defined by a numerical operation, i.e. in our case
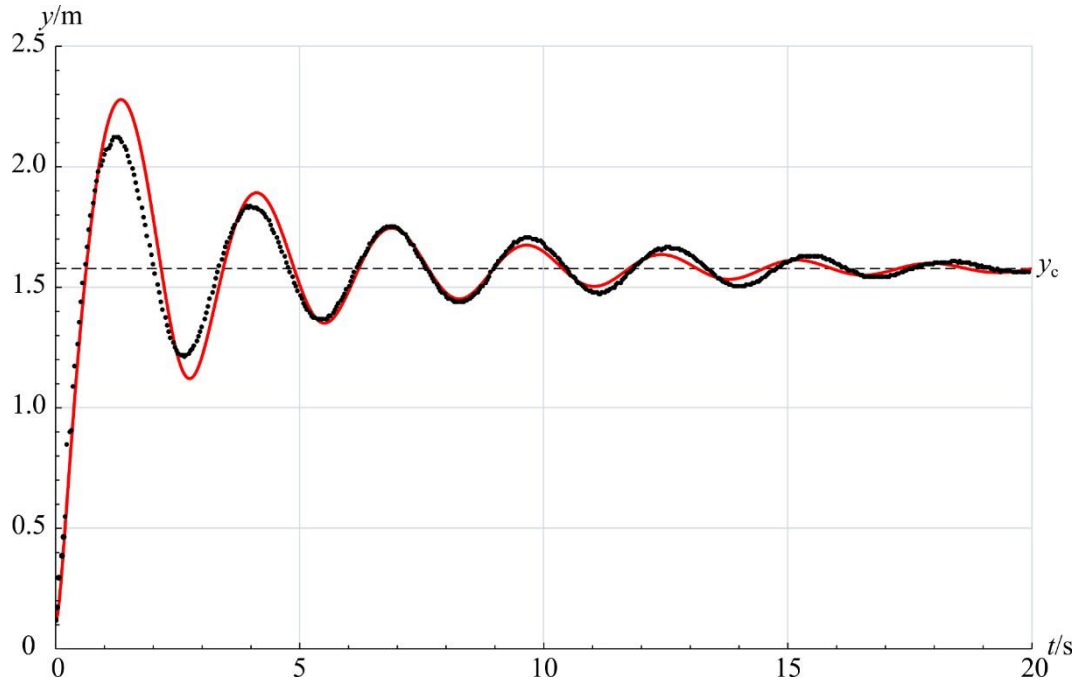
Figure 6. Plots of the video analysis data (black dots) and our more general model fit (the red curve) for $y_c = 1.56$ m, $y_0 = 0.12$ m, $\dot{y}_0 = 0$ m s^{-1}.

numerical solution of the differential equation (6). The results are illustrated in Figure 6. The fitted model (the red curve) corresponding to the equation (6) describes the video analysis data (the black dots) much better that the simplified model in Figure 5. The best fit provides the following values of the model parameters: $y_c = 1.5781(15)$ m, $\alpha = 0.3027(47)$ m and $\beta = 0.627(13)$ m s^{-1}. Following the definition of the parameter α and substituting the balloon weight determined by digital scales $m_b = 3.2(1)$ g as well as the linear density $\eta = 10.40(1)$ g m^{-1}, we get

$$\alpha = \frac{3.2}{10.4}\, \text{m} \approx 0.31\ \text{m},$$

Which corresponds (within about 2.5 %) to the above value provided by the numerical fit.

Certainly, the problem that the balloon and the moving part of the fibre are not perfectly vertical remains, which might be one of the sources of the remaining differences between a fitted model and the video analysis data.

5. Conclusions

Buquoy's problem leads to an interesting example of a 1-D motion with a rich historical context. It represents a specific example of damped oscillations and its solution includes several concepts developed within the introductory physics course: a motion with a variable mass, an effective potential, a solution of a non-linear ODE of the 2nd order requiring computer simulations. Thus, this problem might be considered a little bit more advanced example of an oscillatory motion as well as a specific kind of motion with a variable mass. Our experience proves, that the numerical modelling is accessible to the students and that with an appropriate preparation and support they are usually able to construct a functional numerical model with the help of a suitable software.

The experiment supports other competences like a video analysis of the motion, and the fitting of the theoretical model parameters to the video analysis data. Moreover, the experiment leads to a more general theoretical model that is still accessible to undergraduate students. It demonstrates that the

simplified assumptions – no matter how natural they might seem – are not always realistic; the mass of the balloon as well as the air resistance and friction dumping cannot be neglected.

In our undergraduate physics curriculum, the students meet the Buquoy's problem in three different contexts. Once in classical mechanics as an example of a variable mass problem, when we discuss the equation of motion and the introduction of effective potentials. Then we make use of this problem in a computer supported course concentrated on mathematical background of selected problems from various parts of physics, when the students are programming their own dynamical models and where we also train the fitting of the experimental data. The third instance is when the problem can be used for video analysis within a lab for perspective physics teachers.

We realize that the experimental set still might be improved. Especially for larger oscillations within the first 5 s (see Figure 6), there still is a difference between the fitted model and video data. As already mentioned, one of the problems is to ensure the vertical motion of the balloon and the fibre without any horizontal shifts. The Buquoy's problem is currently solved as a diploma thesis (by a student of a perspective physics teacher's study programme) at our department. The goal is not only to improve the experimental set mentioned above but also to carry out demonstration of this movement.

Nevertheless, we managed to carry out a classroom experimental demonstration of the Buquoy's problem. Within the 2 % uncertainty of the measurement, the experimentally measured values of the maximum height and the parameter of constant mass of the balloon obtained through the numerical fit and calculated from the measured values correspond to each other. The generalized model given by the equation (6) is a more relevant description of an experimental realization of this interesting mechanical problem.

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