The role of mathematics and self-efficacy in learning quantum mechanics

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Abstract.
Quantum mechanics is a highly mathematical topic that many students struggle with. We studied the effect of initial mathematics knowledge and self-efficacy on student learning on an introductory (1st to 2nd year) quantum mechanics course by administering a mathematics pre-test and a self-efficacy questionnaire (N = 50). We correlated the results with course performance (exercise and exam scores). The self-efficacy beliefs of different student groups differed, but correlations between self-efficacy, initial mathematics skills and performance on this course remained modest. In addition, the correlations between self-efficacy beliefs in different aspects of learning quantum mechanics were low to moderate, meaning that students' self-efficacy beliefs in this topic are highly situational. We observed that first-year students with high self-efficacy were more likely to report theoretical physics as their study track. Their initial mathematics skills did not differ from their peers, but they scored slightly better in exercises and the exam than first-year physics students, as the (experimental) physics group also included lower performing individuals. Second-year students who reported physics as their study track scored lower than both first-year physics and theoretical physics students in all measures. The explanation is likely a selective effect: the physics students who take the course in their first year are more comfortable with more advanced physics and mathematics content.

1. Introduction

Quantum mechanics (QM) is a complex and often difficult topic for students of physical sciences. On the other hand, enthusiastic students find studying QM exciting and motivating. In QM, both instructors and students face the difficulty of observing the examined phenomena, and the theory relies strongly on its mathematical constituents. To proceed deeper into the topic, the students need to grasp both the physical interpretation and the mathematical formulations [1, 2, 3]. Mathematically, QM is laid upon the theory of Hilbert spaces, and hence understanding of at least the basic axioms and features of vector spaces is important. This makes linear algebra part of the necessary mathematical toolbox.

On the other hand, mathematics knowledge is hardly the only factor affecting student performance. For example, students' perceptions of themselves as learners influence both their study habits and learning. An important factor in this is self-efficacy, which is defined as "beliefs about their capabilities to produce designated levels of performance that exercise influence over events that affect their lives" [4]. Self-efficacy can be developed mainly by four processes [4]:

(i) Through experiences of mastery and successes.
(ii) Social models (seeing people in similar situations success).

(iii) Social persuasion (verbal prompting).

(iv) Reducing stress and negative feelings.

While the effect on self-efficacy on performance is a widely studied topic, self-efficacy research in physics education tends to focus on introductory courses on mechanics and electromagnetism (see e.g. [5, 6, 7, 8]).

1.1. Introductory QM at University of Helsinki

At the University of Helsinki (UH), the QM curriculum begins in the first or second year of studies, depending on study track. On the first course, *Basics of Quantum Physics*, many students struggle with the mathematics involved. Partly to address this, for the academic year 2017-2018, the curriculum was changed to extend the mathematics course aimed at the students of physical sciences. This change provided more time to address linear algebra. As a result, students attending the Basics of Quantum Physics course in spring 2018 had either taken a non-reformed mathematics course in the previous year, or were taking the new mathematics course simultaneously with the QM course. Some physics minors also attend this course. In most cases they were pre-service teachers who study mathematics as their first subject, meaning they have a strong background in linear algebra. Following a change of the mathematics curriculum at UH, we wanted to study the effect of the varying mathematics knowledge of the students on the learning of QM.

In the study, we needed to address the fact that these student groups are not random. Students who choose to begin their studies in QM in the first year are generally high-achieving. The course is recommended for first-year students who want to choose theoretical physics as their specialization, but the specialization choices are only made official and binding later in the studies. Hence, approximately 50% of first-year physical science students attend Basics of Quantum Physics, but only a fraction of these students give theoretical physics as their specialization.

Basics in Quantum Physics is compulsory also for most other specializations in physical sciences at UH: physics, meteorology and pre-service physics teachers. For these students, the course is recommended in their second year, but as the program is very flexible, a few students push it to their final year of BSc. The students who come to the course in their second or third year generally view themselves as less theoretically inclined than those who take it in their first year, or may be developing a professional identify distinct from experimental or theoretical physics.

To study the selective effect, we wanted to include a self-efficacy survey. As such, we hoped to map the following aspects of learning QM:

- What effect do the previous mathematics studies have on learning in Basics of Quantum Physics?
- What is the role of self-efficacy beliefs in learning outcomes on this course?

2. Curriculum reform

Before the curriculum reform, the mathematics instruction of first year students consisted of two half-term mathematics courses starting from the very beginning of the studies. These courses, *Mathematics for Physicists I-II*, covered the topics needed in the first-year studies of physical sciences: differential, integral and vector calculus, differential equations, series, complex numbers, and linear algebra. As the last topic of the second course, the linear algebra was covered in approximately two weeks. The topics of those two weeks included basics of vector spaces, linear maps, and matrix algebra.

In spring, after the two mathematics courses, approximate half of the first-year students took the Basics of Quantum Physics course. This course covers the basics from the historical background up to a qualitative treatment of the hydrogen atom. The course follows the textbook *Modern Physics* by Randy Harris [9]. The student feedback from past courses suggested that many students felt they had not been taught enough linear algebra in the mathematics courses to grasp the ideas of introductory quantum mechanics.

In the curriculum reform, the first-year mathematics courses aimed at the students of physical sciences were stretched out to three half-terms, and one of the goals was to give more depth to the treatment of linear algebra. Due to this reform, Basics of Quantum Physics and the new *Mathematics for Physicists III* course were taught concurrently. The topics on the new mathematics course do not differ from the ones taught on the unreformed course. However, the basic features of vector spaces, matrix algebra, eigenvalues and eigenvectors, diagonalization, and inverse matrices were given more space on the course. At the same time, the connections between linear algebra and quantum mechanics were emphasized in Basics of Quantum Physics.

3. Methods

On the first lecture of the Basics of Quantum Physics course, we administered a mathematics pre-test using a Moodle-implementation of A System for Teaching and Assessment using a Computer Algebra Kernel (STACK). The pre-test contained six linear algebra problems testing student ability to solve problems on basis vectors, change of basis, inner product, unit vectors, eigenvalues and conjugate transpose. As some students had not yet taken a course on linear algebra, we emphasized that some questions might not be possible to solve and that the test was meant for research purposes and for evaluating the changes in the mathematics courses.

The questions were scored automatically by STACK, and some questions re-scored by hand to allow for a more nuanced partial scoring.

At the end of the course, before the exam, the students were given a course feedback form together with their last exercise sheet. The students get exercise points for giving course feedback, and the feedback is separated from the list of students by an external person before the feedback is delivered to course staff. The questions are answered on a 5-point Likert scale (strongly disagree, 1, to strongly agree, 5). This procedure is familiar to students from other courses.

In this course, five self-efficacy statements were included in the feedback. The self-efficacy statements were adapted from [10] for a QM context. The statements used were:

(i) I understand the concepts of quantum mechanics when I read.
(ii) I understand the concepts of quantum mechanics when I attend a lecture.
(iii) I understand the mathematics used in quantum mechanics.
(iv) My performance in exercises satisfied me.
(v) I believe my exam performance will satisfy me.

In total, 73 students took the mathematics pre-test and 88 feedback and self-efficacy questionnaire. Only data from students who completed both tests and consented to their data being used in research both times is considered, bringing the number of students participating in the research to 50.

For these 50 students, we collected also their exercise and exam scores, and grades. During the course, students were expected to complete weekly homework exercises, which were scored by teaching assistants. Students were encouraged to collaborate in the exercises, and guidance was offered in weekly exercise workshops by teaching assistants. At the end of the course, students had a course exam, to which they were allowed to bring a handwritten sheet of notes. The course grades were formed by a weighed average of the continuous assessment (exercise) grades (33%) and the final exam grade (67%). The lowest passing grade, 1, required 45% of total score and the highest grade, 5, 85%.

All data were pseudonymized before analysis.

4. Results and discussion

The average results for the mathematics pre-test, the self-efficacy questions, and exercise and exam scores are presented in Table 1. The scores for the pre-test ranged from 1.0 to 7.5 out of a maximum of 8. The first-year students, with some notable exceptions, scored fairly well in the first questions, dealing with basis and unit vectors, and mostly empty in the questions concerning matrices. The latter is explained by that they were unlikely to have encountered matrices. On the other hand, older physical science students showed great variance in test scores, with scoring high and some failing even the straightforward vector questions. Also the other science students (including mathematics students) had a large variance in scores, but their average scores were the highest.

Overall self-efficacy was fairly high. Averages for all questions exceeded 3, which corresponds to "neither agree nor disagree", meaning that on average the students agreed that they understood QM concepts when reading, when attending a lecture, they understood the necessary math, they were satisfied with their performance in the exercises and they believed they would be satisfied with their exam results.

The scores for the exercises ranged from 23% to 100% (average 85%), and for the exam from 30% to 100% (average 71%). A detailed breakdown by student groups is presented in Table 1.

4.1. The effect of initial mathematics knowledge

As the student groups are very small after being split by major or specialization, a search for statistically significant differences is not justified. However, there are indications of differences between groups, and we can discuss them qualitatively while keeping in mind the pilot nature of this study.

All first-year students gave their specialization as either physics ($N = 20$) or theoretical physics ($N = 8$), and scored equal in the mathematics pre-test. However, the students who gave theoretical physics as their specialization had slightly higher self-efficacy beliefs and earned higher grades. This does not mean that the students who decide early on that they want to specialize in theoretical physics are the most high-achieving students. As Figure 1 shows, the theoretical physics students scored similar to the high-achieving subset of physics students. The declared specialization may also still change during the BSc studies.

Similarly, as can be seen from Figure 1, there were fewer high-achieving students present among second-year physics students, other physical science students and other science students. Curiously, the second-year physics students achieved lower average points on the mathematics pre-test despite having taken the last part of the non-reformed mathematics course, and, in some cases, additional mathematics courses. They also scored lower in the exam, and, thus, in the grades. This points to a selective effect, as well as differences in studying habits. Second-year students did better in the continuous assessment (exercises) especially when compared to exam performance.

Other physical science and other science majors have, on average, lower grades than the (theoretical) physics majors, but as can be seen from Figure 1, the spread is large. However, other science majors, who are mainly mathematics majors, score highest in the mathematics pre-test.

From the course feedback it is evident that students found the course good and interesting. However, two students brought up the delayed teaching of linear algebra and suggested improvements:

- This course could come after *Mathematics for Physicists III* or the courses could be better coordinated together. At the beginning of the course, I couldn't wrap my head around the mathematics.
- Maybe the usage of operators needs more justification. Where do they come from? Could the course come after *Mathematics for Physicists III*?

Table 1. The number of students N, averages and standard errors of the mean of all student groups for the mathematics pre-test scores, the self-efficacy questions, grades, and exercise and exam scores.

	N	Pre-test	SE reading	SE lecture	SE math	SE exercise	SE exam	Grade	Exercise	Exam
All	50	4.24±0.22	3.9±0.14	3.78±0.15	3.86±0.14	3.78±0.17	3.66±0.15	3.46±0.20	85±3	71±3
Physics 1^{st} yr	20	4.42±0.30	3.9±0.27	3.95±0.21	3.8±0.21	3.3±0.30	3.4±0.23	3.6±0.38	80±5	75±5
Theor. physics	8	4.5±0.49	4.37±0.18	4.37±0.18	4.37±0.18	4.62±0.18	4.25±0.36	4.37±0.32	92±2	87±6
Phys. 2^{nd}+ yr	12	3.91±0.55	4±0.24	3.25±0.42	4.08±0.25	4.16±0.27	3.75±0.27	3.08±0.33	91±4	61±5
Other phys. sci.	6	3.66±0.51	3.33±0.33	3.66±0.33	3.16±0.54	3.5±0.56	3.5±0.42	3±0.51	85±7	60±6
Other science	4	4.62±1.43	3.5±0.5	3.5±0.64	3.5±0.95	3.75±0.75	3.75±0.75	2.75±0.85	83±9	63±13

Table 2. The correlations (r) between the mathematics pre-test scores, self-efficacy questions, grades, exercise and exam scores for all students (N=50).

	Pre-test	SE read	SE lecture	SE math	SE exercise	SE exam	Grade	Exercise pt	Exam pt
Pre-test	1								
SE read	0.200	1							
SE lecture	0.303	0.430	1						
SE math	0.304	0.377	0.453	1					
SE exercise	0.209	0.097	0.292	0.489	1				
SE exam	0.323	0.199	0.497	0.469	0.727	1			
Grade	0.266	0.004	0.368	0.188	0.399	0.402	1		
Exercise pt	0.138	-0.002	0.281	0.112	0.557	0.505	0.524	1	
Exam pt	0.227	-0.006	0.331	0.200	0.292	0.281	0.929	0.265	1

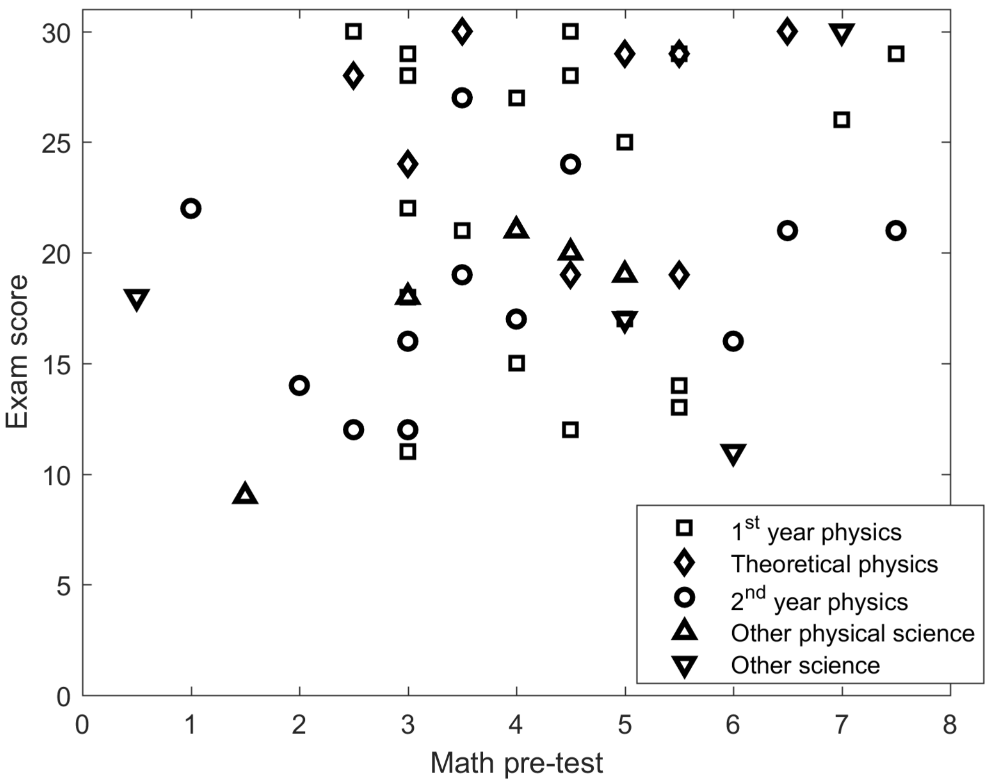

Figure 1. The exam score as a function of the mathematics pre-test.

There is no reason to believe that the performance of students who attended *Mathematics for Physicists III* and *Basics of Quantum Mechanics* at the same time would have been affected by delaying instruction in linear algebra. However, the feelings of the students need to be taken into account, and the topics in the courses can definitely be synchronized to a higher degree in the future.

4.2. The effect of self-efficacy

While the average self-efficacy beliefs are positive, we can see differences between student groups. Second-year physics students scored above first-year physics students in satisfaction in exercise points (they also have higher exercise points) but below in understanding QM through the lectures. Other physical science and other science majors had lower average scores in self-efficacy beliefs in understanding in reading and in mathematics used in quantum mechanics.

All in all, correlations between the mathematics pre-test, self-efficacy questions and course performance measures are mostly weak (see Table 2). The correlations between exercise and exam points and grades naturally result from grades being composed of the exercises and exams. The only strong non-trivial correlation ($r > 0.7$) is found between the students' satisfaction of their exercise performance and their belief they will be satisfied in their exam results ($r = 0.73, r^2 = 0.53$). Interestingly, correlations between the other self-efficacy questions are low ($r < 0.5$). Self-efficacy beliefs are context dependent, but clearly the students perceive their abilities on this course as consisting of very different aspects. For example, the correlations between whether the students feel they understand QM when reading do not correlate with satisfaction in exercises or belief in exam success ($r < 0.2$), and only slightly correlate with their

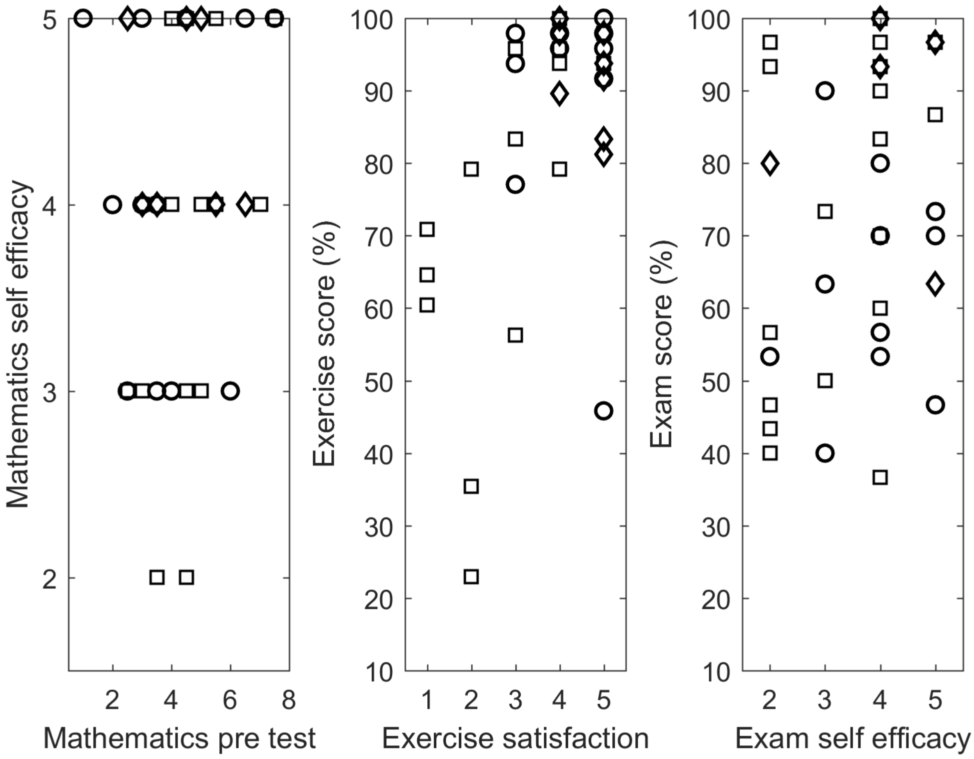

Figure 2. Left: The mathematics self-efficacy beliefs as a function of the pre-test score. Middle: The exercise score as a function of exercise satisfaction. Right: The exam score as a function of self-efficacy beliefs in the exam. Squares: 1^{st} year physics students, diamonds: theoretical physics students, circles: 2^{nd} year physics students.

belief about understanding the mathematics of QM and the lectures ($0.3 < r < 0.5$).

Moderate correlations ($r > 0.5$) appear between accumulated exercise points and satisfaction in exercise performance, and accumulated exercise points and belief in satisfaction of the exam results ($r = 0.56, r^2 = 0.31$ and $r = 0.51, r^2 = 0.26$ respectively). However, the self-efficacy beliefs of exam results and actual points accumulated in the exam correlate very weakly. Exam and exercise points also correlate very weakly. This may point to studying habits: the high exercise averages point to (most) students wanting to accumulate points in the continuous assessment. A more concerning indication is that students may have believed to have learned more than they actually did, making their self-efficacy beliefs about the exam unjustified.

Figure 2 shows the relationship between score in the mathematics pre-test and self-efficacy beliefs in mathematics necessary for QM; the exercise score and satisfaction in exercise and self-efficacy beliefs in exam performance and the exam score. These data also show that the theoretical physics students mostly have high self-efficacy beliefs and mostly do well in the exam, but the correlation between the self-efficacy beliefs and the performance measures are low or non-existent both for the whole student population and the students split by specialization and/or study year.

The lack of correlation between self-efficacy beliefs and the performance measures also means that some students have a high mismatch between their performance and their beliefs about their potential. This is evident in Figure 2 particularly for the exam self-efficacy beliefs and the exam scores, with the presence of students who believed they would be somewhat disappointed in their exam score and got essentially full points in the exam. These students clearly did not have faith in their performance in the exam as it is unlikely they would have been disappointed in their results.

Some students also reported satisfaction in their exercise performance despite a lower than average exercise score and expected to be satisfied in their exam performance but did not achieve a high score in the exam. This latter mismatch is worrying, if the students were expecting to achieve a high grade in the exam and misjudged their abilities. However, not all students aim for the highest grade, and we have not controlled for these expectations.

4.3. Limitations and future prospects

Self-efficacy beliefs are related to specific situations or topics, and they evolve with experiences and discussions [4]. Ideally, we would monitor the development of self-efficacy beliefs by surveys before and after instruction, but in a half-term course, we did not think this feasible. Also, asking about students' self-efficacy beliefs in QM before any formal instruction in the topic would have been difficult to justify. The low correlations between the different self-efficacy beliefs show that the students view their abilities in QM as many-faceted, and we should continue to probe these beliefs and support the development of self-efficacy beliefs.

Mastery of mathematics and self-efficacy are naturally not the only factors that should be considered when studying student learning. Acquiring conceptual understanding in QM is not easy. Conceptual change is difficult to achieve unless it is explicitly addressed, and misconceptions are persistent [11, 12]. While the research on conceptual learning in QM is sparser than in e.g. mechanics, research shows that misconceptions regarding concepts such as stationary states, eigenstates, and time dependence of expectation values are persistent [1, 13, 14].

Course performance does not necessarily reflect the level of learning of the topic, as often assessments focus on solving mathematical problems, i.e. instrumental understanding, and do not measure conceptual and relational understanding [15]. For a more complete view on learning in next year's study, we plan to implement conceptual questions. In the future, we will scrutinize the interplay of initial level of mathematics, mathematics instruction, self-efficacy, and acquired level of conceptual understanding.

5. Conclusions

We have studied the effects of students' initial skills in mathematics on learning the basics of quantum mechanics in conjunction with a change in the timing of mathematics teaching. As the QM course serves several student populations who may differ in skills and interest, we also surveyed the students' self-efficacy beliefs.

We found that the correlations between students' self-efficacy beliefs, initial mathematics level and course performance were generally low. Even the different self-efficacy questions had mostly low correlations with each other, meaning that the students self-efficacy beliefs in QM consist of many different aspects. We found differences between student groups: First-year students who declared theoretical physics as their specialization had higher self-efficacy beliefs on average and somewhat higher course performance, but on an individual level, self-efficacy beliefs did not explain course performance. Second-year physics students had, on average, lower initial mathematics scores than first-year students, despite having had more mathematics studies. This is likely due to a combination of selective effects, declining retention of learned material over time and the timing issues in the mathematics instruction that necessitated the change of curriculum in the first place.

It should be noted that the student groups were small after being split by self-reported specialization. Moreover, these groups are not random, so selective effects are likely to be significant.

Self-efficacy beliefs in general form only a small, albeit important, part in learning. While we did not see a correlation between course performance and self-efficacy beliefs, the connection between self-efficacy beliefs and student specialization should be probed further. To account for other factors in learning quantum mechanics, in the future we will also include conceptual questions.

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