Rapidity evolution of observables at high energies using the gaussian truncation

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Abstract. The quantum-chromo-dynamics of high-energy collisions is effectively described by the colour-glass-condensate. The degrees of freedom in this description are given by path-ordered colour rotations called Wilson lines and their correlators. Their rapidity evolution is given by the JIMWLK equation, which leads to an infinite tower of differential equations. We present a gauge-invariant truncation of this hierarchy in the form of the Gaussian truncation and the machinery for computing the rapidity evolution for several observables with this truncation.

1. Introduction

Today, the biggest predictive uncertainties in the Standard Model arise from theoretical uncertainties in quantum-chromo-dynamics(QCD) contributions to cross-sections measured at high-energy collider experiments. It is via these high-energy experiments that the Higgs-Boson, the last missing piece of the Standard Model, is studied [1]. These high energies also reveal a new state of matter, the quark-gluon-plasma [2] [3], which can give us an insight into the early universe [4]. At high energies, the QCD of particle collisions is well described through the use of the colour-glass condensate. In this domain, the interaction of coloured objects with the CGC medium is well explained through the use of path-ordered colour rotations, called Wilson Lines, as well as their correlators. The rapidity evolution of these correlators is given by the JIMWLK equation [5]. This leads to an infinite hierarchy of coupled differential equations, which are impossible to solve in a closed form and truncations become necessary. The most common truncation relies on the large N_c limit, which is relatively crude and subtly breaks gauge invariance. To get around this, we can perform a gauge invariant truncation of this hierarchy in the form of the Gaussian truncation for the correlators of these Wilson lines. Initial comparison to HERA data for the total and rapidity gap cross-sections show a noticeable improvement in comparison to data which only depend on the dipole correlator [6]. We extend this method to incorporate observables that depend on more complicated correlators and present the machinery for how to compute their rapidity dependence with the Gaussian truncation.

2. The Colour Glass Condensate

The calculations of this proceedings are justified by working in the regime of the Colour Glass Condensate (CGC). To that end, we justify the presence of the CGC and argue the relevant degrees of freedom are given by correlators of path ordered colour rotations called Wilson lines.

Our regime is characterized by high energies and fixed resolution. In the parton model, partons are considered point-like, but their apparent size is given by the resolution of the process they appear in. In other words, in particle collisions, when a probe from the projectile, let's say a photon, interacts with a parton in the target with a large, space-like momentum q, the apparent size of that parton will be given by $Q^2 = -q^2$.

The kinematics of this photon parton interaction must be such that the energy of the collision, given by Mandelstam $\sqrt{s}$, is large. The equation relating these quantities is given by [5]

$$Q^2 = s x_{bj} y, \tag{1}$$

where x_{bj} is Bjorken-x, which has the interpretation of the momentum fraction of the interacting parton to the momentum of the target and y is the inelasticity of the collision, a kinematic variable of $\mathcal{O}(1)$. This equation implies that, at fixed resolution, increases in energy are compensated by decreases in x_{bj}. Furthermore, x_{bj} is related to the rapidity separation between the projectile and target, Y, by $Y = \ln\frac{1}{x_{bj}}$. This implies that a shrinking x_{bj} corresponds to a growing Y and energy, s. Hence, energy evolution can be thought of as rapidity evolution.

Finally, we make reference to the behaviour of the parton distribution functions (PDFs)[7], that suggest that at low x_{bj}, the distribution of a nucleon is dominated by gluons. Altogether, at low x_{bj}, a target nucleus appears to be a Lorentz contracted sheet of soft, interacting gluons that we dub the Colour Glass Condensate(CGC).

3. Wilson lines and their correlators

In the context of a hard projectile interacting with a target consisting of soft gluons, the leading contribution to this process will be a colour rotation of the projectile that depends on the path of the projectile. We call this colour rotation a Wilson line. More precisely, this kinematic regime allows us to use the Eikonal approximation and replace infinite soft gluon exchanges of an incoming parton with a Wilson line that follows the classical trajectory of said parton, the full chain of arguments of which can be found in [8].

Wilson lines enter cross-sections through Wilson line correlators [9] and are the objects we would compute in order to test the CGC and calculate observables. The most common and simplest example of a Wilson line correlator is that corresponding to a quark anti-quark dipole, dubbed the dipole correlator:

$$\langle \mathrm{Tr}(U_{\boldsymbol{x}} U_{\boldsymbol{y}}^{\dagger}) \rangle, \tag{2}$$

where $U_{\boldsymbol{x}}$ is the Wilson line of the quark and $U_{\boldsymbol{y}}^{\dagger}$ of the anti-quark respectively. $\boldsymbol{x}$ and $\boldsymbol{y}$ are the transverse coordinates, the positions where the target is pierced, of the two respective Wilson lines.

There is no analytic expression for these correlators, however their evolution is given by the JIMWLK equation [5]. As an example, we consider the rapidity evolution of the dipole correlator:

$$\frac{d}{dY}\langle \mathrm{Tr} U_{\boldsymbol{x}} U_{\boldsymbol{y}}^{\dagger}\rangle_Y = \frac{\alpha_s}{\pi^2}\int d^2z \mathcal{K}_{\boldsymbol{xzy}} \left(\langle [\tilde{U}_{\boldsymbol{z}}]^{ab} \mathrm{Tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}^{\dagger})\rangle - C_F \langle \mathrm{Tr} U_{\boldsymbol{x}} U_{\boldsymbol{y}}^{\dagger}\rangle_Y \right). \tag{3}$$

Present are the strong coupling constant, α_s, integral kernel, $\mathcal{K}_{\boldsymbol{xzy}}$, a Wilson line for a gluon, $[\tilde{U}_{\boldsymbol{z}}]^{ab}$ and the Casamir, C_F. For more details on what these objects are and why they appear, read [5] for a full overview. The salient feature to be noted from eq(3) is that the rapidity derivative of the 2-point correlator is expressed in terms of the 3-point correlator, the first term in the right-hand-side of eq(3). When one tries to determine the rapidity evolution of the 3-point correlator, they would find that it depends on the 4-point correlator. This process repeats ad infinitum leading to an infinite tower of differential equations. Since there is no known way of solving this entire tower directly to more than leading order, one needs to truncate this process in some manner in order calculate anything useful. That's where the Gaussian truncation comes in.

4. Gaussian Truncation

The Gaussian truncation is a special case of a more generic truncation of the JIMWLK hierarchy that preserves gauge invariance and gives one approximate access to all Wilson line correlators. Just how good an approximation this is, is still under investigation, but these are advantages not afforded by more common truncations, such as the large N_c approximation. The first step in performing the Gaussian truncation is by reparametrizing the rapidity dependence of the Wilson line correlator into an operator that acts on an initial condition. Begin by considering the rapidity evolution of an arbitrary Wilson line correlator:

$$\frac{d}{dY}\langle A\rangle(Y) = \underbrace{(\frac{d}{dY}\langle A\rangle_Y)\langle A\rangle_Y^{-1}}_{-M_Y}\langle A\rangle_Y, \tag{4}$$

which permits a trivial path-ordered exponential solution,

$$\langle A\rangle_Y = Pe^{-\int_{Y_0}^{Y} dY' M(Y')}\langle A\rangle_{Y_0}. \tag{5}$$

This has shifted the rapidity dependence onto M. For correlators more complex than the dipole, care must be taken to perform evolution in a symmetric manner. Since we do not know the exact form of M, we write it in the most generic form we are able to that preserves the structure and properties of the Wilson line correlator it acts upon. By structure, we mean that if we consider a family of Wilson line correlators that map from singlets of n quarks and n anti-quarks, this operator should be an endomorphism on this set. The most important property that needs to be preserved is that of the coincidence limits. Hence, we introduce the basic building block of such an operator:

$$i\overline{\nabla}^a_x U_y = -\delta^{(2)}_{xy} U_y t^a. \tag{6}$$

This functional derivative operator can be shown to commute with the Wilson line correlator and instead act on the gluon distribution [5]

$$\int D[U](i\overline{\nabla}A)Z_Y[U] = \int D[U]A(i\overline{\nabla}Z_Y[U]). \tag{7}$$

It is this property of $i\overline{\nabla}$ that implies that the functional derivative of A must preserve the structure and properties of A, since the only thing that changes is the distribution we average over. We use this functional derivative to build a generic operator

$$\begin{aligned}&- M_Y\langle A\rangle_{Y_0} =: \\ &\langle \Bigg[\frac{1}{2!}\int_{u_1u_2} G_{Y,u_1u_2} i\overline{\nabla}^a_{u_1} i\overline{\nabla}^a_{u_2} + \\ &\frac{1}{3!}\int_{u_1u_2u_3} G^a_{Y,u_1u_2u_3} K_a^{a_1a_2a_3} i\overline{\nabla}^{a_1}_{u_1} i\overline{\nabla}^{a_2}_{u_2} i\overline{\nabla}^{a_3}_{u_3} + \cdots\Bigg] A\rangle_{Y_0}.\end{aligned} \tag{8}$$

This has shifted the rapidity dependence on to the G's, which are defined such that this produces the correct evolution. In the literature, the G's carry the interpretation of gluon exchange functions [10]. K stands for all the possible colour structures and ensures there are no free colour indices remaining. The Gaussian truncation is preformed by truncating this operator at the first term

$$-M_Y\langle A\rangle_{Y_0} =: \langle \left[\frac{1}{2!}\int_{u_1u_2} G_{Y,u_1u_2} i\overline{\nabla}^a_{u_1} i\overline{\nabla}^a_{u_2}\right] A\rangle_{Y_0}. \tag{9}$$

With this operator, we may now write any Wilson line correlator in terms of these G's. For example, the dipole correlator becomes

$$\langle \mathrm{Tr}(U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}})\rangle_Y = N_c e^{-C_f \mathcal{G}_{Y,\boldsymbol{xy}}}\langle \mathrm{Tr}(U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}})\rangle_{Y_0}, \tag{10}$$

where

$$\mathcal{G}_{Y,\boldsymbol{xy}} := \int^Y dY' \left(G_{Y',\boldsymbol{xy}} - \frac{1}{2}(G_{Y',\boldsymbol{xx}} + G_{Y',\boldsymbol{yy}})\right). \tag{11}$$

This Gaussian truncation operator can be applied to any Wilson line correlator to get an expression in terms of these G's. All that remains is to determine how these G's behave, as they know carry all the rapidity and position content. This can be done simply, however, by applying the Gaussian truncation to both sides eq(3). This gives us a closed differential equation in G that can be solved numerically for its rapidity dependence [11]. Thus we have constructed an operator that translates any Wilson line correlator into an expression involving gluon exchange functions whose behaviour we know. We are now in a position to actually compute observables.

5. Observables

There are many observables that become computable once one knows how to calculate Wilson line correlators. We mention two here. The first is a dijet process[12] of quark→quark + photon

$$\frac{d\sigma^{qA\to q\gamma X}}{d^3k_1 d^3k_2} \sim \left[\langle tr(U_b U^\dagger_{b'})\rangle + \langle tr(U_v U^\dagger_{v'})\rangle - \langle tr(U_b U^\dagger_{v'})\rangle - \langle tr(U_v U^\dagger_{b'})\rangle\right], \tag{12}$$

which depends on four dipole correlators. The second observable is that of the gluon spectrum given by [13]

$$\frac{d\sigma(qA\to qgX)}{d\sigma qA\to qX} = \frac{2C_R\alpha_s}{\pi^2}\frac{\int_{\boldsymbol{r}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\int_{\boldsymbol{x}}\frac{\boldsymbol{x}\cdot(\boldsymbol{x}+\boldsymbol{r})}{\boldsymbol{x}^2(\boldsymbol{x}+\boldsymbol{r})^2}\left\{\langle tr(U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{x}+\boldsymbol{r}})\rangle - \langle[\tilde{U_{\boldsymbol{x}+\boldsymbol{r}}}]^{ab}\mathrm{Tr}(t^a U_{\boldsymbol{0}} t^b U^\dagger_{\alpha\boldsymbol{x}+\boldsymbol{r}})\rangle\right\}}{\int_{\boldsymbol{r}} e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\langle tr(U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{x}+\boldsymbol{r}})\rangle}. \tag{13}$$

Eq(13) is the ratio of two cross-sections, one with gluon emission to the inclusive one without. This calculation is important as it is part of the observable that computes the energy loss of partons due to the presence of cold nuclear matter. This is an important consideration if one wants to compute the energy loss due to hot nuclear matter, as in the case of trying to probe the quark gluon plasma.

6. Numerics

The previous sections describe all one needs in order to numerically compute Wilson line correlators, and the observables that depend on them. We present two examples of numeric calculations below, but let it be emphasized that these are exploratory, proof of concept results. There has been no paramter fitting or matching of initial conditions performed yet. In the first example we demonstrate what such numerics would give us for the 3-point correlator.

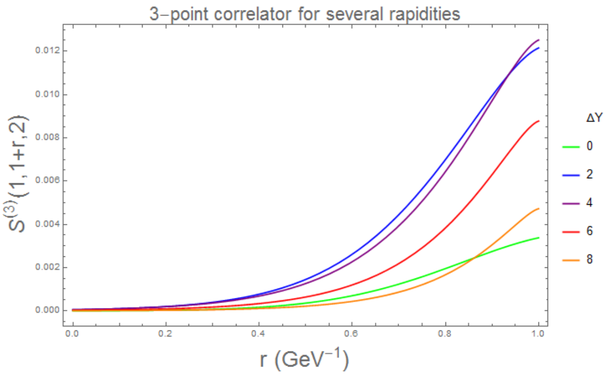

Figure 1. A plot of the 3-point correlator, $S^{(3)}(1,1,+r,2) := \langle [\tilde{U}_{\boldsymbol{x}+\boldsymbol{r}}]^{ab}\mathrm{Tr}(t^a U_{\boldsymbol{0}} t^b U^\dagger_{\alpha\boldsymbol{x}+\boldsymbol{r}})\rangle$, as a function of one of its coordinates as it varies from one coincidence to another. This is done for multiple rapidities, all a ΔY from a reference rapidity that sets the initial condition. Credit to Javier Albacete for providing numerics for gluon exchange functions.

This correlator and the 2-point correlator enter into the induced gluon spectrum eq(13), which we present an exploratory calculation of the induced gluon spectrum below.

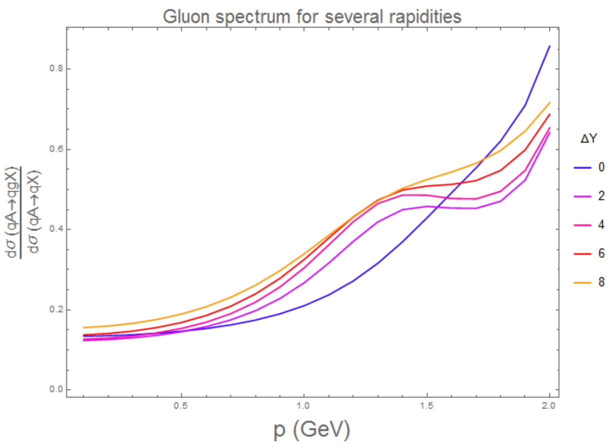

Figure 2. The gluon spectrum - the ratio of a gluon emission cross-section to the inclusive cross-section without emission - as a function of momentum. This is done for multiple rapidities, all a ΔY from a reference rapidity that sets the initial condition. Credit to Javier Albacete for providing numerics for gluon exchange functions.

7. Discussion

We motivated that we work in a regime that is relevant for studying exciting subjects like the Higgs Boson and the QGP. This is the regime of the CGC where the relevant degree of freedom are given by Wilson line correlators. Through the use of the Gaussian Truncation any Wilson line correlator can reparamterised in terms of gluon exchange functions. This parametrisation preserves the structure and the coincidence limits of the correlators. The gluon exchange functions can be numerically solved for from the JIMWLK equation. Altogether this allows one to numerically compute observables that depend on Wilson line correlators, such as certain dijet processes and the induced gluon spectrum. In this proceedings, only observables that depend on simple correlators were discussed for brevity, but the Gaussian truncation does provide access to all n-point correlators and can be used to compute observables that depend on them, such as the other jet processes in [12].

One thing to be determined is what information is gained by the Gaussian truncation over other techniques beyond its analytic properties. Comparison of the dipole to experiment shows a better fit than the large N_c limit truncation, but it is not know how well the Gaussian truncation of higher n-point correlators compares. On the other hand, it needs to be determined what information is lost when one uses the Gaussian truncation and how this can be amended if we include higher order terms than the Gaussian term.

8. Outlook

Having established a method to calculate a number of interesting observables, what remains to be done is to carry out these calculations and compare with data. There are many properties of the Gaussian truncation to be explored in this manner; How well it compares with data, to what kinematic limits can we extend this process, how important is each term in the parametrisation of the rapidity evolution, to which degree are corrections beyond the Gaussian term are necessary.

The Gaussian truncation is a novel numerical tool with far reaching properties, but much remains to be seen on how useful it truly is.

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