Entanglement and gravity

JM Hartman[1], SH Connell[1], C Engelbrecht[1] and F Petruccione[2]

1. University of Johannesburg, Johannesburg, South Africa
2. University of Kwa-Zulu Natal, South Africa

E-mail: hartman.jonathan2@gmail.com

Abstract. Entanglement in Quantum Mechanics leads to a non-local correlation between two particles. The question arises as to whether changes in local gravity, or the equivalently the local background metric affects the correlation. There are several approaches to answering this question. This work discusses the treatment of the background gravity by using the equivalence principle to map it to a local acceleration. One can then use the concept of Rindler frames to treat the problem in the context of special relativity for uniform acceleration. More general accelerations are also considered, including those which relate directly to motion as a result of a force, such as gravity. This is a more realistic scenario. A Thomas precession is shown to manifest for the particle spins. The description of the effect of this on the maximal violation of a Bell inequality has been written down for a range of scenarios for systems of entangled particle pairs. At least in principle, it is established that a gravity can effect entanglement and also how this could be quantified.

1. Introduction

Quantum entanglement is a phenomenon in quantum mechanics whereby two particles are correlated in such a way that when the state of one is measured, then the state of the other is always known (in the case of a maximally entangled system). The concept goes back to the famous EPR paradox, initially put forth by Einstein, and it set off a flurry of debates over it's nature. The ideas as to it's nature, and that of the EPR paradox, only became empirically testable, however, in 1964 when John Bell published a paper on an argument now known as Bell's theorem. In the paper, Bell derived an inequality, Bell's inequality, which would hold if the correlations could be described by classical local realism. In essence, Bell's theorem says that no physical theory that incorporates local realism can reproduce all the predictions of quantum mechanics as quantum mechanics itself predicts a violation of Bell's inequality. Bell calculated this for the case of non-relativistic quantum mechanics though and it's only recently that people have started investigating how Bell's inequality changes when relativistic effects are considered.

The first to investigate a relativistic Bell inequality was Czacor in 1997 [1], who found that the relativistic effects of entangled massive particles affect the correlations between them in the special case where both are moving in the same direction. The origin of this deviation from the non-relativistic Bell's inequality is the length contraction when measuring the spin correlations in the laboratory frame, though Bell's theorem is still valid as the maximal violation of Bell's inequality is still obtained when the spins are measured with respect to a basis which has a compensating rotated direction.

Later, in 2003, Terashima and Ueda [2] investigated a similar situation to Czacor but from the perspective of an observer moving perpendicular to the motion of the entangled particles, moving in opposite directions, with opposite spins. Like, Czacor, they found that there would be a degradation of the entanglement correlations as opposed to an observer that is in the frame of the source. They similarly also concluded that maximal violation of Bell's inequality could still be obtained when the spins are measured in in a basis with compensating directions. In 2004, Lee and Chang-Young [3] combined Czacor's result with Terashima and Ueda's to derive a more general case for the special relativistic Bell theorem.

Later, some authors investigated quantum communication in accelerated frames [4] [5], and others investigated quantum entanglement in curved space-times [6] [7].

In our own previous work [8], we have explored the effect of accelerated motion on entanglement in a relativistic kinematical context. This current work takes the matter further, exploring this effect for accelerations as a result of central potentials such as gravity.

2. The Bell inequality and Thomas precession

Consider a source of entangled particles of opposite spins with detectors on either side that measures their spins in the longitudinal direction of their spin orientations. If local realism is assumed then the Bell inequality is given by

$$\left|P\left(\vec{a},\vec{b}\right) - P\left(\vec{a},\vec{c}\right)\right| \leqslant 1 + P\left(\vec{b},\vec{c}\right), \tag{1}$$

where $\vec{a}$ and $\vec{b}$ are unit vectors of the detector settings (defining the basis) for measuring the spin projections, $\vec{c}$ is any other arbitrary unit vector and $P\left(\vec{a},\vec{b}\right)$ etc are the average products of the results of the spin projections as measured by the detectors. It can easily be shown that the quantum mechanics violates this inequality. For example, given that quantum mechanics predicts that $P\left(\vec{a},\vec{b}\right) = -\vec{a}\cdot\vec{b}$, then if we take $\vec{a}$ and $\vec{b}$ to perpendicular to each other and $\vec{c}$ to be at a 45 degree angle to each, then $P\left(\vec{a},\vec{b}\right) = 0$ and $P\left(\vec{a},\vec{c}\right) = P\left(\vec{b},\vec{c}\right) = -0.707$, giving the left hand side of (1) as 0.707. As the right hand side evlaluates to $1 - 0.707 = 0.293$, we have a violation of the Bell inequality.

The original formulation of the Bell inequality is not practical for experiment, however, as it is restricted to the special case in which outcomes of the spin measurements are always anti-correlated whenever the detectors are parallel. A more general inequality that was more suitable to experiment was given in 1969 by Clauser, Horn, Shimony and Holt. It is now called the CHSH inequality [9] and is given by

$$C(a,a',b,b') \equiv \left\langle \hat{a}\otimes\hat{b}\right\rangle + \left\langle \hat{a}\otimes\hat{b}'\right\rangle + \left\langle \hat{a}'\otimes\hat{b}\right\rangle - \left\langle \hat{a}'\otimes\hat{b}'\right\rangle \leqslant 2, \tag{2}$$

Here, $\vec{a}$ and $\vec{a}'$ are two detector settings on side A and $\vec{b}$ and $\vec{b}'$ are two detector settings on side B. This is violated in non-relativistic quantum mechanics since quantum mechanics predicts a maximal violation as

$$C(a,a',b,b') = 2\sqrt{2}. \tag{3}$$

Now let's look at the relativistic case where two particles are moving apart from each other in their combined rest frame, while an observer is moving perpendicular to the two particles. This could correspond, for example, to an observer noting the decay of a moving parent particle into two daughters. As said before, this situation was investigated by Terashima and Ueda [2]. The adding of two velocities (the motion of the particle and the motion of the observer) together in special relativity is not commutative. In fact, when you combine two boosts together that are not parallel, you do not simply get a new boost in a new direction but a boost plus a rotation. This rotation as observed in the lab frame with respect to the rest frame of the decaying particle, or the centre of mass frame (CoM) is called the Wigner angle. This Wigner angle showed up in Terashima and Ueda's calculation of the maximal violation of Bell's inequality as

$$C(a,a',b,b') = 2\sqrt{2}\cos^2\delta, \tag{4}$$

where the Wigner angle, δ, in the case of a perpendicular boost is given by

$$\tan\delta = \frac{\sinh\xi \sinh\chi}{\cosh\xi + \cosh\chi} \tag{5}$$

where ξ and χ are the rapidities of the particles in the CoM frame and the observer frame with respect to the CoM frame respectively given by $\frac{v_1}{c} = \tanh\xi$ and $\frac{v_2}{c} = \tanh\chi$.

We have previously shown [8] that if the two particles are accelerating away from each other, then the Wigner angle δ as seen in the moving frame can be said to be changing with respect to time, so that it looks like a precession. This precession is called the Thomas precession. By considering an acceleration as a sequencing of infinitesimal changes in velocity, we have shown that the Thomas precession can be written as

$$\tan(\delta_0 + \Delta\delta(t)) = \frac{\sinh(\xi + \Delta\xi(t))\sinh\chi}{\cosh(\xi + \Delta\xi(t)) + \cosh\chi} \quad \text{with} \quad \Delta\xi(t) = \int_0^\tau a_P(\tau)d\tau. \tag{6}$$

This links the acceleration a explicitly to the change in the rapidity $\Delta\xi(t)$ and hence to the Thomas precession. Here a_P and τ are the proper acceleration and proper time.

The acceleration induced Thomas precession would have the effect of gradually weakening the violation of the Bell or CHSH inequality. One can imagine however that the unit vectors describing the orientation of the detectors can be adjusted to compensate the effect of the Thomas precession, so that the maximal violation can be restored. The acceleration is the result of a force. In this way an entanglement witness (the CHSH, or some other incarnation) is sensitive to the force environment for the decaying system. Put alternatively, entanglement can be used to measure the force (for example local gravity) of the entangled system.

We now turn to how one could apply this to, in principle, the use entanglement to detect small forces between particles.

3. Thomas precession for the Newtonian gravitational force between two entangled particles

In order to consider the two body problem, let's consider the that the decay products are of different masses m_1 and m_2 and velocities, v_{p1} and v_{p2} with respect to the lab frame. In this case, the CoM frame is the one where the sum of the momenta of the two particles is zero, or

$$\vec{p}_1 = m_1\vec{v}_{p1} = -m_2\vec{v}_{p2} = -\vec{p}_2 = \mu\vec{v}_1, \tag{7}$$

where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is called the reduced mass and $\vec{v}_1 = \vec{v}_{p1} - \vec{v}_{p2}$ is the relative velocity between the particles. This reduces the number of variables so that we can treat it as a one-particle system. Let's also assume that there is a gravitational attraction between the two particles. Then if we assume conservation of mechanical energy, we have

$$E = K + V = \frac{1}{2}\mu {v_1}^2 - \frac{Gm_1m_2}{r}, \tag{8}$$

for the kinetic energy and the gravitational potential energy and where r is the distance between the two particles and G is Newton's gravitational constant. Therefore, the relative speed v_1 is given by

$$v_1 = \sqrt{\frac{2\left(Gm_1m_2 + Er\right)}{\mu r}}. \tag{9}$$

Referring back to equations 5 and 6, we find that the Thomas precession is given by

$$\tan\left(\delta_0 + \triangle\delta\right) = \frac{v_1 v_2}{c\left(\sqrt{c^2 - {v_1}^2} + \sqrt{c^2 - {v_2}^2}\right)}, \tag{10}$$

by applying the hyperbolic identities $\sinh(\text{artanh } x) = \frac{x}{\sqrt{1-x^2}}$ and $\cosh(\text{artanh } x) = \frac{1}{\sqrt{1-x^2}}$ and where v_2 is the speed of the lab frame with respect to the CoM frame. Substituting (9) into (10) gives

$$\tanh(\delta_0 + \triangle\delta) = \frac{v_2\sqrt{2(Gm_1m_2 + Er)}}{c\left(\sqrt{c^2\mu r - 2(Gm_1m_2 + Er)} + \sqrt{\mu r(c^2 - v_2{}^2)}\right)}. \tag{11}$$

So substituting (11) into (4) provides a possible application of quantum entanglement and the relativistic Bell inequality to detect a small gravitational force between two particles. If this were possible, it would be useful because normally gravity as a force is too weak to detect on the scale of individual particles. One of the reasons one would want to do this is that some theories of quantum gravity predict a deviation from the standard inverse square law of Newton's law of gravity. For example, in string theory, if large extra dimensions are present, then the gravitational potential would be modified as (see [10] and [11])

$$V_g = -\frac{\Gamma\left(\frac{d-2}{2}\right) l^{d-3} G\mu}{\pi^{\frac{d-2}{2}} r^{d-2}}, \tag{12}$$

where d is the total number of spacial dimensions and l is the compactification size of the extra dimensions. There is a problem with the above equations though, in that this potential for Newtonian gravity is non-relativistic. One would have to apply general relativity, or some approximation of general relativity. We tackle this in the next section.

4. Calculation for the relativistic gravitational force between two entangled particles

Whenever one has to go beyond the Newtonian gravitational force to make it compatible with special relativity, one has to use Einsteins theory of general relativity instead (the most accurate theory known concerning non-quantum gravity). General relativity claims that the gravitational force is not a "force" as such but a curvature in the space-time background generated by matter and energy. The relationship between mass-energy and the amount of curvature is given by a tensor equation called Einstein's field equation which related a measure of the space-time curvature to the stress-energy tensor (the tensor which encodes the matter content of the equation). For our purposes we do not need to use the full Einstein field equation though. Just note that Newton's force law can be recovered from it if we assume that the gravitational field strength is very weak and when the speed of the source is much slower than that of light. Interestingly, if we make the same approximation that the gravitational field is very weak but we allow for relativistic speeds then we obtain an equation very similar to the Lorentz force law for electromagnetism instead. At this level there are some similarities between the gravitational force and the electromagnetic force, [12]

$$\vec{a} = -\vec{E}_g - 4\vec{v} \times \vec{B}_g, \tag{13}$$

where E_g and B_g have the same definitions as the electric and magnetic fields in electromagetism, $\vec{E}_g = -\vec{\nabla}\phi$ and $B_g = \nabla \times \vec{A}$. From the general solution of the linearised approximation of Einstein's field equation, the scalar and vector potentials are given by

$$\phi = \frac{G}{c^2}\int \frac{\rho_0}{|\vec{x} - \vec{x}'|}\frac{1}{(1 - \frac{v_s{}^2}{c^2})^{\frac{1}{2}}}dx'^3 \tag{14}$$

and

$$\vec{A} = \frac{G}{c^4}\int \frac{\rho_0\vec{v}_s}{|\vec{x} - \vec{x}'|}\frac{1}{(1 - \frac{v_s{}^2}{c^2})^{\frac{1}{2}}}dx'^3, \tag{15}$$

where $\vec{v}_s$ is the velocity of the source and ρ_0 is the mass density of the source. For a point mass, we can take

$$\rho_0 = m\delta(\vec{x}'), \tag{16}$$

where $\delta(\vec{x}')$ is the Dirac delta function. Substituting (16) into (14) and (15), we get

$$\phi = \frac{1}{c^2\left(1-\frac{v_s{}^2}{c^2}\right)^{\frac{1}{2}}} \quad \text{and} \quad \vec{A} = \frac{1}{c^4\left(1-\frac{v_s{}^2}{c^2}\right)^{\frac{1}{2}}}\frac{Gm\vec{v}_s}{r}. \tag{17}$$

Substituting these back into equation (13) and utilising the definitions of E_g and B_g, we get

$$\vec{a} = -\frac{1}{\left(1-\frac{v_s{}^2}{c^2}\right)^{\frac{1}{2}}}\frac{Gm}{r^2}\hat{r} \quad \text{or} \quad \vec{a} = -\frac{G}{r^2}\left(\frac{m_1}{\left(1-\frac{v_1{}^2}{c^2}\right)^{\frac{1}{2}}} + \frac{m_2}{\left(1-\frac{v_2{}^2}{c^2}\right)^{\frac{1}{2}}}\right)\hat{r}. \tag{18}$$

for the case of our two particle system.

This expression for the acceleration can then be deployed in equation (6) to determine the Thomas precession as a result of the treatment of gravity at this mostly classical level of approximation.

5. Quantum mechanical description of the gravitational force

We can also model the decay into two particles in a gravitational field in quantum mechanics by modelling it as a scattering solution similar to Coulomb scattering, given the similarities of Newton's law of gravitation to Coulomb's force law in electrostatics. In the quantum mechanical description of the two particle system, the Hamiltonian in the CoM frame is given by

$$\hat{H} = \frac{1}{2\mu}\hat{p}^2 + V(\vec{r}) \quad \text{with} \quad V(\vec{r}) = -\frac{Gm_1m_2}{\mu r} = -\frac{G(m_1+m_2)}{r} \tag{19}$$

for the gravitational potential so that

$$\hat{H}\psi = -\frac{1}{2\mu}\hbar^2\nabla^2\psi - \frac{G(m_1+m_2)}{r}\psi. \tag{20}$$

The solution to Schrödinger's equation in (20) is given by

$$\psi = \exp(ikr\cos\theta)\ {}_1F_1\left(in', 1, ikr\left(1-\cos\theta\right)\right), \tag{21}$$

where ${}_1F_1(a,b,c)$ is the hyper-confluent geometric function and $n' = \frac{Gm_1m_2}{\hbar^2 k}$. It would be useful to compare our result to the Rutherford scattering cross section far away from the "scattering centre" in order to validate our numerical results. We plotted our scattering solution on the same axes as that of Rutherford scattering on a logarithmic scale in figure (1). Since we for our purposes we are interested in small radii, we want to see if it approaches the Rutherford scattering at large radii. We see that it does according to the right part of figure (1). The procedure is then to extract the expectation value for the acceleration a from this solution (21) in the standard way, and then deploy this as well in the calculation of the Thomas precession.

6. Conclusion

We have reviewed some literature on relativistic entanglement and the Bell-type inequalities. In the second section on Bell's theorem, we used the work on relativistic entanglement by Terashima and Ueda [2] to show how one could use entanglement to detect small accelerations between particles by measuring the Thomas precession. In the following section we developed the idea further and suggested how one could use the result to detect weak forces between entangled

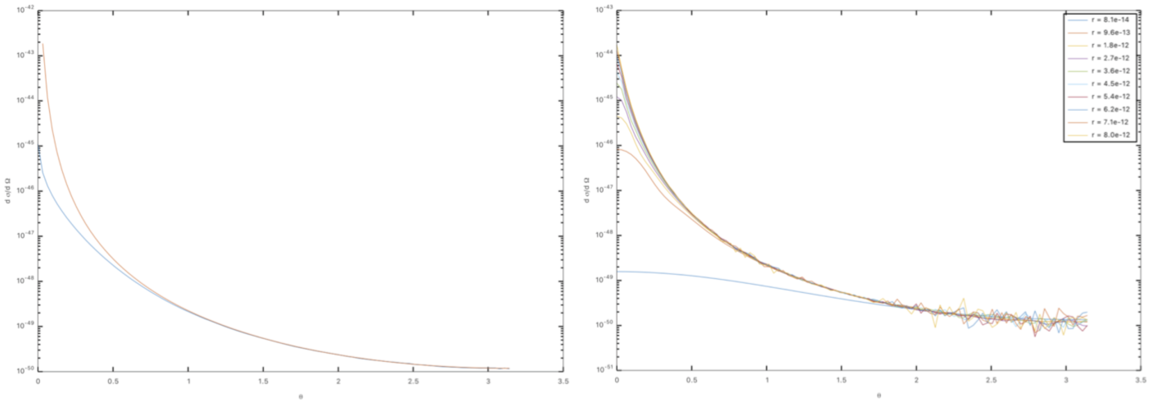

Figure 1. Left : Scattering cross section compared with Rutherford scattering. Right : Comparison of the cross sections at different distances between the particles.

particles and explained how it could be extended to also detect large extra dimensions. The calculation of the Thomas precession required the calculation of the change in rapidity of the particles in the CoM frame. Given that Newton's law of gravitation is not compatible with special relativity, the next section was devoted to a relativistic description of the gravitational force between the two particles, utilising the weak field approximation of general relativity. This section showed explicitly at a mostly classical level of approximation how the desired rapidity could be calculated. A subsequent section introduced a non-relativistic quantum approach to calculating the same rapidity. Future work would extend the calculations to more accurate treatments. In particular, a more accurate treatment should be given to the relativistic scenario. It would also evaluate the size of the effects in a range of plausible scenarios. Given that previous lower limits for extra dimensions are measured down to nanometre scale, an experiment as we have described, if successful, could could constrain it even further to the femtometre scale.

7. Acknowledgements

The authors acknowledge financial support of the National Research Foundation (NRF) as well as provision of accommodation by Resolution Circle.

References

[1] Czachor M 1997 *Phys. Rev. A* **55**(1) 72–77
[2] Terashima H and Ueda M 2003 *Quantum Information & Computation* **3** 224–228
[3] Lee D and Chang-Young E 2004 *New Journal of Physics* **6** 67
[4] Van Enk S J and Rudolph T 2003 *Quantum Info. Comput.* **3** 423–430 ISSN 1533-7146
[5] Alsing P M and Milburn G J 2003 *Phys. Rev. Lett.* **91**(18) 180404
[6] Terashima H and Ueda M 2004 *Phys. Rev. A* **69**(3) 032113
[7] Ralph T C, Milburn G J and Downes T 2009 *Phys. Rev. A* **79**(2) 022121
[8] Hartman J M, Connell S and Petruccione F 2015 *Proceedings of the 60th Annual Conference of the South African Institute of Physics (SAIP2015)* ed Chitambo M and Venter A pp 499–504
[9] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23**(15) 880–884 URL `https://link.aps.org/doi/10.1103/PhysRevLett.23.880`
[10] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 *Physics Letters B* **429** 263 – 272 ISSN 0370-2693
[11] Zwiebach B 2004 *A first course in string theory* (Cambridge: Cambridge Univ. Press) URL `https://cds.cern.ch/record/789942`
[12] Wald R M 1984 *General Relativity* 1st ed (University Of Chicago Press) ISBN 0226870332