Quantum Boltzmann evolution of the Quark-Gluon Plasma

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Abstract. The rapid equilibration of the Quark-Gluon Plasma, produced in nucleus-nucleus collisions in a far-from-equilibrium initial state, seems to be difficult to understand theoretically. One reason could be that almost all existing approaches based on the relativistic Boltzmann equation neglect quantum-statistics features of the quarks and gluons. Against this background, we put forward a novel Monte-Carlo method to solve the Boltzmann equation, with quantum effects included.

1. Introduction

Recent heavy-ion experiments have confirmed the existence of a deconfined plasma of quarks and gluons, the Quark Gluon Plasma (QGP) [1, 2]. The rapid (≤ 1 fm) onset of hydrodynamical flow of the QGP requires rapid thermalization of the plasma; this is contrasted with the Color Glass Condensate (CGC) picture which is a model of the initial conditions where the incoming nuclei are saturated with gluons up to a saturation scale Q_s. How this rapid thermalization of the QGP occurs is a matter of much theoretical interest. It has been suggested by [3] that if the initial conditions are dense enough, and number changing processes in early times suppressed enough, that a transient Bose-Einstein condensate could develop. The development of a condensate could significantly influence the dynamics of the plasma [4]. Whether or not the condensate is kinetically able to form in the time constraints is currently unknown. Initial applications of kinetic theory have neglected quantum effects, these classical models not only do not thermalize rapidly enough, but also drive the system to a Maxwell-Boltzmann type equilibrium, precluding the formation of a condensate. We thus want to study the kinetic theory of the QGP without neglecting quantum effects.

2. The Boltzmann equation

To study the thermalization of the QGP and see if the rapid formation of a Bose-Einstein condensate is kinetically allowed we turn to the Boltzmann equation. The Boltzmann equation is a way of describing the evolution of the distribution function of a non-equilibrium system, and reads,

$$\frac{d}{dt}f(\boldsymbol{x},\boldsymbol{p},t) = \mathcal{C}[f(\boldsymbol{p},t)] - \boldsymbol{v}\nabla f(\boldsymbol{x},\boldsymbol{p},t), \tag{1}$$

where $f(\boldsymbol{x},\boldsymbol{p},t)$ is the distribution function. The $\boldsymbol{v}\nabla f(\boldsymbol{x},\boldsymbol{p},t)$ term describes the flow of matter through space, for the purposes of this paper we will assume a spatially isotropic distribution

function,and thus this term shall be zero.We will further assume momentum isotropy so that the distribution function is only a function of the magnitude of the momentum. $\mathcal{C}[f(\boldsymbol{p}),t]$ is the so-called collision term which is a functional that describes the interaction between different particles, for bosons,for a binary elastic interaction this term reads

$$\begin{aligned}\mathcal{C}[f(\boldsymbol{p},t)] =&\frac{1}{2E_p}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{1}{\nu}\int\frac{d^3p_3}{(2\pi)^3 2E_3}\int\frac{d^3p_4}{(2\pi)^3 2E_4}|\mathcal{M}_{p_1p_2\to p_3p_4}|^2\\ &\times[f(p_3)f(p_4)(1+f(p))(1+f(p_2))-f(p)f(p_2)(1+f(p_3))(1+f(p_4))]\\ &\times(2\pi)^4\delta^4(p+p_2-p_3-p_4).\end{aligned} \tag{2}$$

This expression details the rate at which particles are scattered into, and out of a particular momentum state, by summing every kinetically allowed scattering event weighted by the quantum mechanical amplitude ($\mathcal{M}^2$) for such a particular scattering to occur and the Bose enhancement factors (the $(1+f)$ terms). The Bose enhancement factors are absent in the classical formulation. Most existing approaches, like [5] perform a classical approximation.

It is readily checked that the Bose-Einstein distribution,

$$f_{\mathrm{BE}}(p)=\frac{1}{e^{\beta E(p)}-1}, \tag{3}$$

satisfies

$$\mathcal{C}[f_{\mathrm{BE}}(p)]=0, \tag{4}$$

and thus is a fixed point of the Boltzmann equation.

The Bose enhancement terms strictly increase the rate at which scatterings occur, which could lead to a more rapid thermalization.

The Boltzmann equation is a non-linear functional equation and has no general analytic solution, and we resort to either applying simplifying assumptions to the model we study or rely on a numerical solution which we detail now.

3. Numerical solution of the Boltzmann equation

To solve the Boltzmann equation, we use a Monte Carlo algorithm inspired by [5], we improve upon in by not making the aforementioned approximation, albeit using a different method. This was done by [6] as well, although using a different, more computationally expensive method than what we will perform.

The idea behind our algorithm is as follows, the Boltzmann equation describes the scattering of particles into new momentum states, so once an initial distribution is specified, we draw an ensemble of N particles from this distribution, calculate the probability that each unique particle pair will interact in some timestep Δt, $\left(\text{there are } \frac{N(N-1)}{2}\text{such pairs}\right)$, "roll the dice" to determine which particle pairs interact and update the momentum of the scattered particle. This specifies a new ensemble which is a representation of the new distribution function, at $t+\Delta t$.

An ensemble of finitely many samples is going to have fluctuations, possibly large, which limits the quality of the approximation of the distribution function, this can be overcome by averaging over ensembles, the fluctuations average out, and we can extract a better approximation of the distribution function.

With this in mind, instead of only performing the scattering step once, we roll the dice multiple times to get many different ensemble representations of the new distribution function, which we can reconstruct from the ensembles. We detail how we reconstruct the distribution function is presented in the following section.

The details of how the interaction probability of a pair of particles is calculated can be found here [6].

Position space is discretized into cells, only particles within a single cell can interact. We've only considered a single cell at this point. Having multiple cells will allow us to reintroduce the flow term into the Boltzmann equation.

3.1. Reconstruction of the distribution function

Our algorithm requires the ability to reconstruct a distribution function from a given ensemble.

A typical technique to extract a probability distribution from an ensemble is to bin the data into a histogram and interpolate an approximation of the function. This approach is, however, not desirable in this context as the reconstructed distribution function should conserve energy and accurately replicate the slope of, in particular, the Bose-Einstein distribution at small momenta (where the contribution from Bose enhancement is expected to be largest). This approach was taken by [6], and they required a very large number of particles work.

Instead, we smear each particle in the ensemble into a Gaussian-like "basis" function(chosen to satisfy the criteria above), and the combination of these basis functions gives an approximation of the true distribution function. This procedure also has a physical justification in the uncertainty principle, where the width of the basis function corresponds to the uncertainty.

We found the most effective radial distribution functions given by the following form,

$$R(p;\mu,\sigma) = \frac{2}{\sigma\sqrt{\pi}}\mathrm{Erf}\left(\frac{\mu}{\sigma}\right)\exp\left(-\frac{p^2+\mu^2}{\sigma^2}\right)\sinh\left(\frac{2p\mu}{\sigma^2}\right), \tag{5}$$

where the parameters σ and μ control the width and the peak of the curve respectively. Energy conservation constrains one of the parameters, we specify the width based upon the average separation between particles in momentum space, which specifies the value of the peak.

Once the width σ is specified the position of the peak, μ, is specified(by energy conservation) by

$$\frac{\mu}{\mathrm{Erf}\left(\frac{\mu}{\sigma}\right)} = E \tag{6}$$

when smearing a particle with energy E.

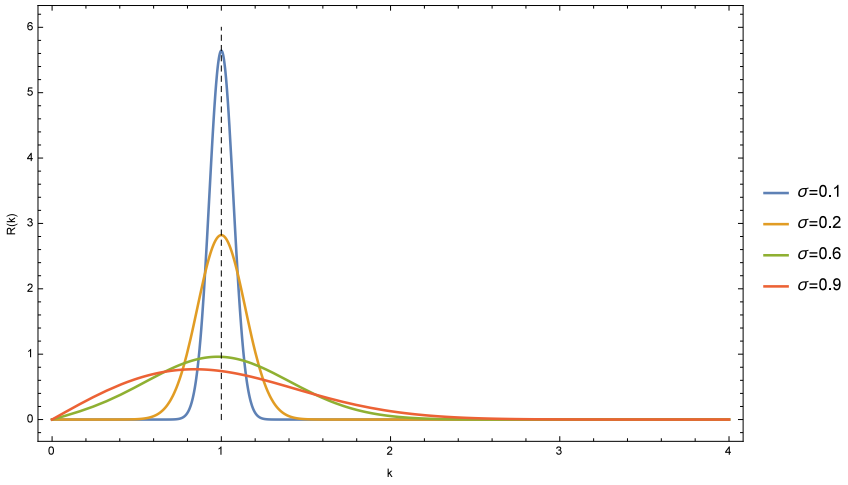

Figure 1: Here we see how the basis functions look for various values of the width parameter, the original particle had a momentum specified by the dashed line.

Here we show a couple of examples of reconstructions done by this technique,

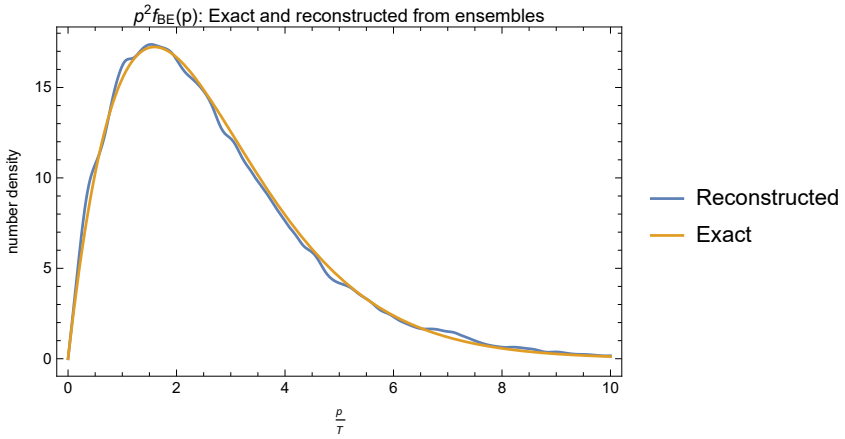

Figure 2: A reconstruction of the radial Bose-Einstein distribution, the reconstruction was done with 500 ensembles of 64 particles.

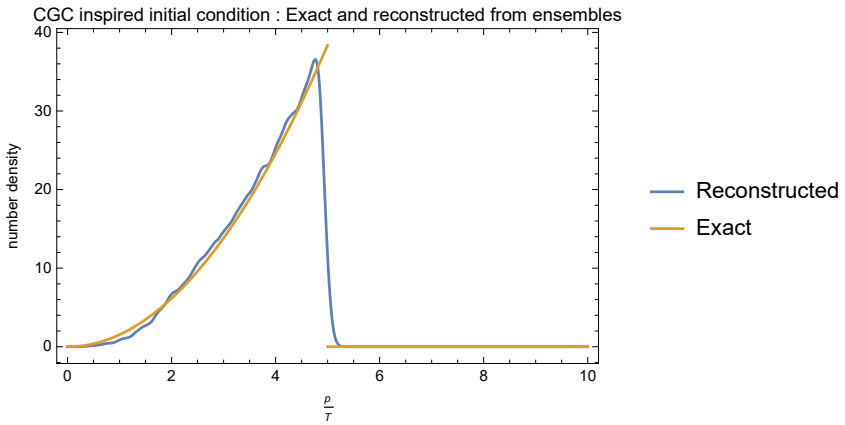

Figure 3: A reconstruction of a radial Fermi-like distribution(inspired by the CGC), the reconstruction was done with 500 ensembles of 64 particles.

this shows both the ability to accurately reconstruct the Bose-Einstein distribution, especially getting the small momentum behaviour correct, and that it generalises to another distribution, the same parameters are used for the reconstruction of both distributions.

4. Preliminary results

Whilst a full simulation has not yet been carried out, we present some preliminary results. We considered the effect that adding quantum fluctuations would have on the scattering rate of binary gluons, in particular the dominant t-channel exchange process,

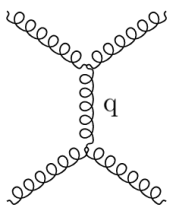

Figure 4: Feynman diagram for t-channel binary gluon scattering

with the corresponding regulated cross-section

$$\frac{d\sigma}{dt} = \frac{9\pi\alpha_s}{(t - m_D^2)^2} \tag{7}$$

We compute the interaction likelihoods of this process on an ensemble of 64 particles drawn from a Bose-Einstein distribution for both the quantum case(inclusion of Bose enhancement) and the classical case(no Bose enhancement)

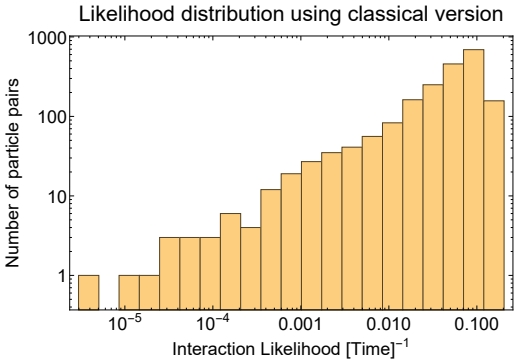

Figure 5: Calculation of the interaction likelihoods 100 particle pairs using the classical algorithm, the scattering time will be dominated by the largest interaction likelihood, in this case $0.1s^{-1}$

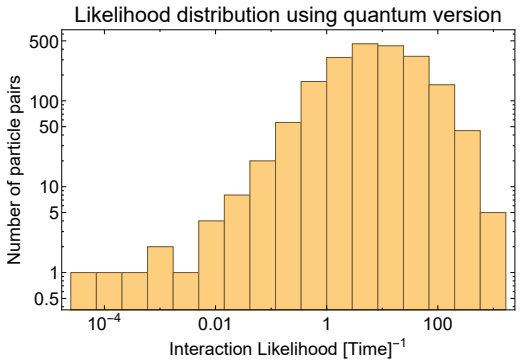

Figure 6: Calculation of the interaction likelihoods 100 particle pairs using the quantum algorithm, the scattering time will be dominated by the largest interaction likelihood, in this case $100s^{-1}$

By looking at the most likely interactions in each case, we see an increase in the interaction rate in the quantum case by 2-3 orders of magnitude. This is a promising result, whether it is enough to see rapid thermalization is yet to be seen.

5. Conclusion

We develop a numerical algorithm to solve the Boltzmann equation with quantum effects included; this required the development of a novel way of reconstructing a distribution function from a representative ensemble. We hope that this can be used in the near future to investigate the kinetics of Bose-Einstein condensate formation in the QGP. The high interaction rates generated when including quantum effects requires a very short timestep, so detailed studies using this code have not yet been done. The are some obvious numerical optimisations to be done, for instance, the parallelization of the calculation of interaction probabilities. The next features to be developed would be a robust technique for detecting the presence of a condensate, an implementation of a running coupling, the inclusion of number changing processes and the reintroduction of the flow term.

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