Quantum Corrections to the Kink-Antikink Potential

Zander Lee
Supervised by: Prof. H. Weigel

Stellenbosch University

5 July 2016

Introduction

Non-linear field theories may produce classical solutions that have a spatially localized energy density.

- These solutions are called solitons or solitary waves.
- They maintain their localization over time.
- Technically, solitons should retain shape and velocity after collisions (otherwise they are solitary waves).
- These properties allow for an extended-particle interpretation.

It is relatively straightforward to calculate the classical energy of these configurations, however vacuum polarization effects may significantly alter classical predictions.

- Calculate the vacuum polarization energy (VPE) of the kink-antikink potential in the ϕ^4 model.

VPE and the spectral method

The vacuum polarization energy (VPE):

- The leading, *i.e.* one-loop, quantum correction to the energy of a static soliton.
- Renormalized sum of the shifts of the zero point energies of the quantum fluctuations due to their interaction with the background configuration generated by the soliton.

We want to calculate the VPE from scattering data in the framework of spectral methods.

VPE and the spectral method

Consider the Lagrangian density with background potential $V(\varphi)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\,. \tag{1}$$

- $\exists$ a static solution to the field equation, $\phi_0(x)$, with fluctuations $\eta(x,t)$, i.e. $\varphi(x,t) = \phi_0(x) + \eta(x,t)$.
- The fluctuations $\eta(x,t) = \eta_\omega(x)e^{-i\omega t}$ are subject to a relativistic wave-equation

$$\omega^2\eta_\omega(x) = \left[-\frac{d^2}{dx^2} + U(\phi_0(x))\right]\eta_\omega(x)\,. \tag{2}$$

- $U(\phi_0(x))$ is the potential for the fluctuations obtained by expanding the full wave-equation about $\phi_0(x)$ to linear order in $\eta(x,t)$.

VPE and the spectral method

For the fluctuations $\eta_\omega(x)$ we have

$$E_{vac} \sim \frac{1}{2}\sum_j \left(\omega_j - \omega_j^{(0)}\right). \tag{3}$$

- Difference in vacuum energy between the full and free theories.
- Energy of an infinite set of harmonic oscillators.

Problems:

- The sum is not finite.
- The sum is not really a sum, since the system may have a continuum of scattering states.

VPE and the spectral method

The background polarizes the quantum fluctuations $\eta_\omega(x)$ in two aspects:

- It creates bound states with energies ω_j,
- It distorts the density of the scattering states.

To account for both bound states and the continuum of scattering states, the VPE is given by

$$E_{vac} = \frac{1}{2}\sum_{j}^{\text{b.s.}} \omega_j + \frac{1}{2}\int_0^\infty dk\, \omega \Delta\rho(k)\Big|_{\text{renorm.}} \tag{4}$$

where $\omega^2 = k^2 + m^2$ (with k being the momentum and m the mass of the fluctuating field) and $\rho(k)$ is the density of states.

VPE and the spectral method

It can be shown that

$$\Delta\rho(k) = \frac{1}{\pi}\frac{d\delta(k)}{dk}. \qquad (5)$$

In one space dimensions there are two scattering channels when the potential is reflection invariant, the symmetric ($p = +$) and antisymmetric ($p = -$) channel.

This leads to

$$E_{vac} = \frac{1}{2}\sum_{j}^{\text{b.s.}} \omega_j + \sum_{p=\pm}\int_0^\infty \frac{dk}{2\pi}\,\omega\frac{d\delta_p(k)}{dk}\bigg|_{\text{renorm.}} . \qquad (6)$$

VPE and the spectral method

Renormalization is accomplished in two steps:

- The divergent contributions to the momentum integral in are identified from the Born series.
- They are subtracted from the integral and added back in as Feynman diagrams.
- The divergences of the Feynman diagrams are removed with the help of standard counterterms whose coefficients are universal for a fixed renormalization scheme.
- For the current problem this procedure is quite simple because only the first order tadpole diagram is divergent.
- This diagram is local and can be fully removed under renormalization.

VPE and the spectral method

Levinson's theorem:

$$\delta_+(0) = \left(n_+ - \frac{1}{2}\right)\pi \qquad \text{and} \qquad \delta_-(0) = n_-\pi\,, \tag{7}$$

where n_+ and n_- are the number of symmetric and antisymmetric bound states, with threshold states ($k = 0$) only contributing $1/2$.

Applying Levinson's theorem and integrating by parts yields

$$E_{vac} = \frac{1}{2}\sum_j (\omega_j - m) - \sum_{p=\pm}\int_0^\infty \frac{dk}{2\pi}\,\frac{k}{\omega}\left[\delta_p(k) - \delta_p^{(1)}(k)\right]\,, \tag{8}$$

where $\delta_p^{(1)}(k)$ is the Born approximation to the phase shift in channel p.

Kink-antikink configuration

Lagrangian density of ϕ^4 model:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{2\lambda}\right)^2 . \tag{9}$$

Classical static solutions to the Euler-Lagrange equation:

$$\phi_0 = \pm\frac{m}{\sqrt{2\lambda}}, \quad \phi_{\mathrm{K}}(x) = \frac{m}{\sqrt{2\lambda}}\tanh\left(\frac{mx}{2}\right) = \phi_{\bar{\mathrm{K}}}(-x) . \tag{10}$$

- Static kink solution: connects the two vacuum solutions between $x = \pm\infty$ (particle).
- Antikink solution: spatial reflection of kink (antiparticle).

Total (classical) kink energy:

$$E_{\mathrm{cl}} = \int_{-\infty}^{\infty} dx\,\epsilon(x) = \int_{-\infty}^{\infty} dx\,\frac{m^4}{8\lambda}\,\mathrm{sech}^4\left(\frac{mx}{2}\right) = \frac{m^3}{3\lambda} . \tag{11}$$

Kink-antikink configuration

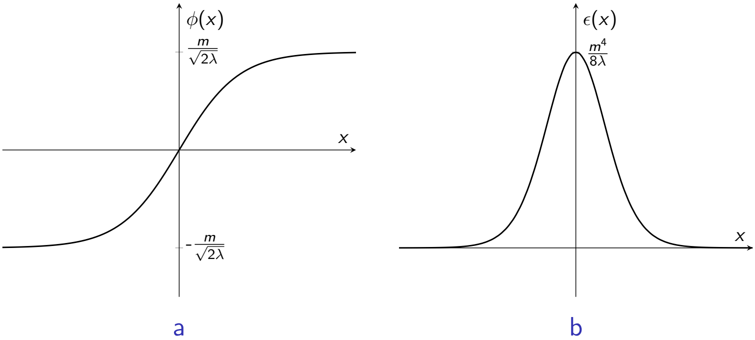

Figure : (a) A schematic plot of the static kink solution. (b) The energy density of the kink. It is localized, with a width characterised by $1/m$.

Kink-antikink configuration

The field configuration that describes a kink and antikink at fixed separation $2R$ reads

$$\phi_R(x) = \frac{m}{\sqrt{2\lambda}}\left[\tanh\left(\frac{m}{2}(x-R)\right) - \tanh\left(\frac{m}{2}(x+R)\right) + 1\right]. \quad (12)$$

- Due to the non-linear structure this is not a solution to the field equation, unless widely separated.

This has the classical contribution to the energy:

$$V_{\mathrm{cl}}(R) = \frac{2m^3}{\lambda}\left[Rm + \frac{3}{\tanh(Rm)} - \frac{2+3Rm}{\tanh^2(Rm)} + \frac{2Rm}{\tanh^3(Rm)} - 1\right]. \quad (13)$$

Kink-antikink configuration

To compute the VPE contribution we need to solve the wave-equation

$$k^2\eta_\omega(x) = \left[-\frac{d^2}{dx^2} + u(x)\right]\eta_\omega(x)\,, \tag{14}$$

where

$$u(x,R) = U(\phi_R(x)) - m^2 = 3\lambda\left[\phi_R(x)^2 - \frac{m^2}{2\lambda}\right]\,, \tag{15}$$

is the potential induced by the kink-antikink background.

- The VPE has the interpretation of a quantum correction to the potential for particle-antiparticle interaction.

Kink-antikink configuration

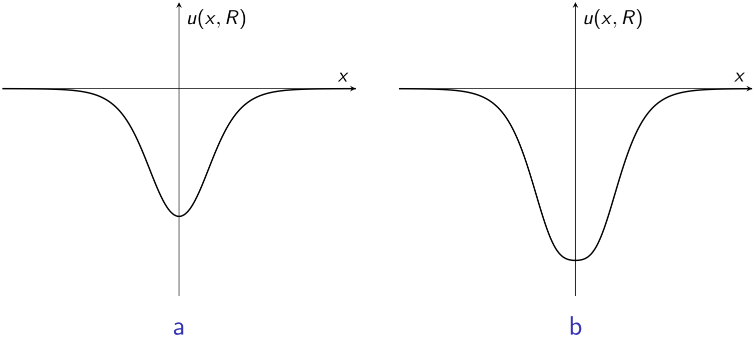

Figure : A schematic plot of the potential for the fluctuations induced by the kink-antikink background, for two different values of the separation distance R, (a) $R = \frac{1}{2}$ (b) $R = 1$.

Kink-antikink configuration

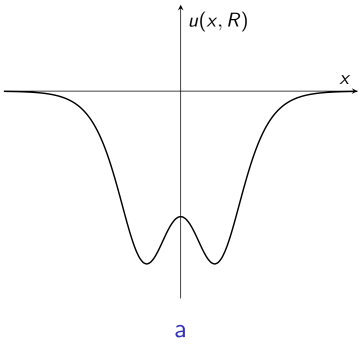

Figure : A schematic plot of the potential for the fluctuations induced by the kink-antikink background, for two different values of the separation distance R, (a) $R = 2$ (b) $R = 4$.

Kink-antikink configuration

- When the kink and antikink are widely separated ($R \to \infty$), each possesses the translational zero mode $\omega^2 = 0$.
- When reducing R, these two translational modes mix with one turning negative (imaginary energy eigenvalue).
- This must be avoided in order to obtain a well-defined quantum theory for the fluctuations.

Since the distance R is fixed, no fluctuations in this direction should be admitted, which induces the constraint

$$\int_{-\infty}^{\infty} dx\, \eta_\omega(x) z(x) = 0\,, \tag{16}$$

where

$$z(x) = N \frac{d}{dR}\phi_R(x) \quad \text{with} \quad N^{-2} = \int_{-\infty}^{\infty} dx \left(\frac{d}{dR}\phi_R(x)\right)^2 . \tag{17}$$

Kink-antikink configuration

This constraint turns the wave-equation into an integro-differential equation

$$-\eta''_\omega(x) = k^2\eta_\omega(x) - u(x)\eta_\omega(x) + \alpha\, z(x)\,, \tag{18}$$

where

$$\alpha = \int_{-\infty}^{\infty} dy\ \left[z(y)\sigma(y) - z''(y)\right]\eta_\omega(y)\,. \tag{19}$$

Since $z(x)$ vanishes asymptotically, this equation represents a well-defined scattering problem from which phase shifts and their Born approximations can be computed.

Note that the constraint only affects the symmetric $(p = +)$ channel.

Kink-antikink configuration

Bound states (antisymmetric channel):

- Construct a set of basis states by implementing boundary conditions on the non-interacting solutions at a large distance L.
- Compute matrix elements of the operator

$$-\frac{d^2}{dx^2} + U(\phi_R(x)), \tag{20}$$

  and find the eigenvalues.
- The eigenvalues below m^2 are the bound state energies (squared).
- Since the bound state wave-functions decay exponentially at large x, the corresponding energies are not sensitive to the value of L (for sufficiently large L).

Kink-antikink configuration

For the symmetric channel with the constraint:

- Diagonalize the projector $\mathbb{1} - |z\rangle\langle z|$.
- Decouple the unstable mode from the spectrum.
- Diagonalize the operator

$$-\frac{d^2}{dx^2} + U(\phi_R(x)), \tag{21}$$

  in the resulting basis.
- Note that this always causes a zero mode to appear in the symmetric channel, since the decoupled mode must be counted a zero eigenvalue bound state.

Scattering data

For the contribution from the continuum of scattering states, we need to calculate the phase shift. In the symmetric channel we parameterize:

$$\eta_\omega(x) = \mathrm{e}^{\nu_S(x)} \cos[kx + \delta_S(x)]\,, \tag{22}$$

with the two functions being related by (variable phase approach)

$$\cos[kx + \delta_S(x)] \frac{d\nu_S(x)}{dx} = \sin[kx + \delta_S(x)] \frac{d\delta_S(x)}{dx}\,. \tag{23}$$

The physical phase shift δ_+ is related to $\delta_S(x)$ via

$$\delta_+ = \lim_{x\to\infty} \delta_S(x)\,. \tag{24}$$

Scattering data

Substituting this into our wave-equation and simplifying yields the differential equations

$$\frac{d\delta_S(x)}{dx} = -\frac{1}{k}c(x)\left[u(x)c(x) - \alpha z(x)e^{-\nu_S(x)}\right], \qquad (25)$$

$$\frac{d\nu_S(x)}{dx} = -\frac{1}{k}s(x)\left[u(x)c(x) - \alpha z(x)e^{-\nu_S(x)}\right], \qquad (26)$$

where α is a Lagrange multiplier and

$$c(x) = \cos[kx + \delta_S(x)], \qquad s(x) = \sin[kx + \delta_S(x)], \qquad (27)$$

with the initial conditions $\nu_S(0) = 0$ and $\delta_S(0) = 0$ and the constraint

$$\int_0^\infty dx\, z(x)c(x)e^{\nu_S(x)} = 0\,. \qquad (28)$$

Scattering data

Numerically integrate E_{vac} from k_{min} to k_{max}, for each value of k:

- Take a guess for the Lagrange multiplier α.
- Solve the differential equations for $\delta_S(x)$ and $\nu_S(x)$ using adaptive-step Runge-Kutta method from x_{min} to x_{max}.
- Check if the constraint is satisfied:
  - If not, use root-finding algorithm to update α.
  - If yes, set $\delta_+(k) = \delta_S(x_{max})$ and move on to the next k.

For the antisymmetric channel we derive the differential equation

$$\frac{d\delta_A(x)}{dx} = -\frac{u(x)}{k}\sin^2(kx + \delta_A(x))\,, \tag{29}$$

and solve it using the same procedure above (without α).

Results

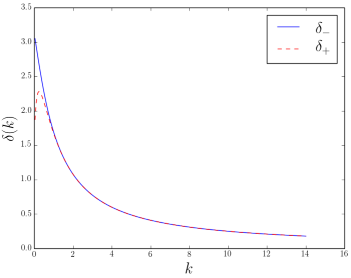

Figure : Phase shift in the symmetric (δ_+) and antisymmetric (δ_-) channels for $R = 0.25$ with $m = 2$.

bound state	$R = 0.25$
$\omega_1^{(+)}$	0
$\omega_1^{(-)}$	1.815

Table : Symmetric ($\omega_j^{(+)}$) and antisymmetric ($\omega_j^{(-)}$) bound states, including the zero mode with $m = 2$.

Results

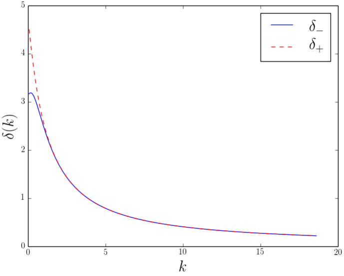

Figure : Phase shift in the symmetric (δ_+) and antisymmetric (δ_-) channels for $R = 0.5$ with $m = 2$.

bound state	$R = 0.5$
$\omega_1^{(+)}$	0
$\omega_2^{(+)}$	1.905
$\omega_1^{(-)}$	1.350

Table : Symmetric ($\omega_j^{(+)}$) and antisymmetric ($\omega_j^{(-)}$) bound states, including the zero mode with $m = 2$.

Results

Table : Symmetric ($\omega_j^{(+)}$) and antisymmetric ($\omega_j^{(-)}$) bound states for various values of the separation distance R with $m = 2$. For large R, $\omega_1^{(-)}$ turns into a zero mode and $\omega_2^{(\pm)}$ approach the breather mode $\sqrt{3}m/2$.

bound state	$R = 0.75$	$R = 1.0$	$R = 3.0$	$R = 4.0$
$\omega_1^{(+)}$	0	0	0	0
$\omega_2^{(+)}$	1.719	1.592	1.723	1.731
$\omega_1^{(-)}$	0.927	0.606	0.012	0.002
$\omega_2^{(-)}$	1.999	1.955	1.740	1.733

Results

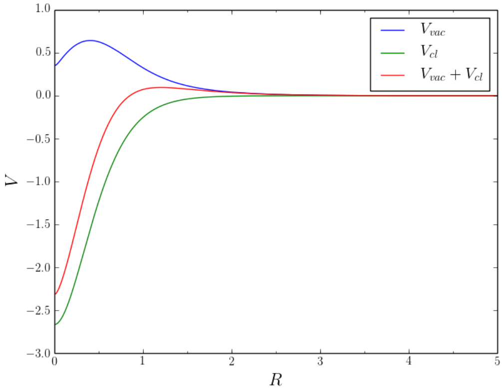

Figure : $V_{vac} = E_{vac} - 2E_{kink}$ as function of the separation distance R with $m = 2$. Also shown is the classical potential V_{cl}.

Conclusion and open problems

- We have estimated the one-loop quantum correction to the kink-antikink potential.
- We observe that the quantum correction mitigates the strong attraction seen in the classical kink-antikink potential.
- Our findings are consistent with the VPE being just a correction to the classical energy.
- The quantum correction produces a mild repulsion at intermediate separation suggesting that these corrections stabilize a classically unstable configuration.

Conclusion and open problems

- We found that $E_{vac}(R = 0) \neq 0$, which is unexpected.
- We will attempt to resolved this discrepancy by calculating the contribution from the source $J(x)$, which comes from the fact that $\phi_R(x)$ is not a static solution of the wave-equation.
- There may be both, a direct contribution and an indirect one as a modification of the constraint may turn out inevitable.
- We still need to perform these calculations for the sine-Gordon model, once the above issues are resolved.

Acknowledgements

UNIVERSITEIT
STELLENBOSCH
UNIVERSITY

NITheP
National Institute for
Theoretical Physics

References I

- R. Rajaraman, *Solitons and Instantons*, North Holland, 1982.
- T. H. R. Skyrme Int. J. Mod. Phys. **A3** (1988) 2745. Article reconstructed by I. Aitchison.
- G. S. Adkins, C. R. Nappi, E. Witten, Nucl. Phys. **B228** (1983) 552.
- H. Weigel, Lect. Notes Phys. **743** (2008) 1.
- N. Graham, M. Quandt, H. Weigel, Lect. Notes Phys. **777** (2009) 1.
- R. G. Newton, *Scattering Theory of Waves and Particles*. Springer, New York, 1982.

References II

- T. Sugiyama, Prog. Theor. Phys. **61** (1979) 1550.
- F. Calegero, *Variable Phase Approach to Potential Scattering*. Acad. Press, New York and London, 1967.
- N. Graham, R. L. Jaffe, Phys. Lett. B **435** (1998) 145.
- N. Graham, R. L. Jaffe, M. Quandt, H. Weigel, Annals Phys. **293** (2001) 240.
- Z. Q. Ma, A. Y. Dai, J. Phys. A **21**, 2085 (1988).
- D. T. J. Feist, P. H. C. Lau, N. S. Manton, Phys. Rev. D **87** (2013) 085034.