

Quasinormal modes for a spin-3/2 field in the Reissner-Nordström black hole background

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Abstract. We will present a quasi-normal modes (QNMs) calculation for a scalar (spin-0) field in a Schwarzschild black hole background and comment on how this could be generalised to QNMs for a Weyl field, as well as fields in a Reissner-Nordström black hole background. These are the first steps towards calculating the QNMs for the spin-3/2 field in a Reissner-Nordström black hole background, which is the ultimate aim of my current research project. We shall make use of the Wentzel-Kramers-Brillouin (WKB) approximation method to computing the QNMs for black holes perturbed by fields, working to high orders of approximation. We shall work to sixth order for the systems described above.

1. Introduction

Black holes can be excited by perturbations, which are due to interactions with fields. The excitation causes oscillations in the spacetime around the black hole. These oscillations are exponentially damped and their temporal dependence is equivalent to their energy loss to spatial infinity, thus the name “quasinormal modes”. The quasinormal frequencies (QNFs) associated with these modes are complex, with the real part describing how damped the modes are, and the imaginary part describing the rate at which the modes decay. It has been shown that one can study the thermodynamics of a black hole through the understanding of QNMs, see Ref. [1]. In this work we are interested in building up methods to compute the quasinormal modes of a spin-3/2 field in a Reissner-Nordström background, as an extension to the work done on the QNMs for a spin-3/2 in an D -dimensional Schwarzschild black hole in [2]. The calculations for QNMs involve solving a master equation (which is Schrödinger-like) of the type

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0, \tag{1}$$

which is the radial part of the overall wavefunction describing some field in a certain spacetime background. The method which can be used to solve this master equation is a semi-analytical approximation method called the WKB method, which involves the patching of solutions at some classical turning points. There are other approximation methods that one can use to solve Eq. (1), such as the Asymptotic Iteration Method (AIM), see Ref [3]. All calculations in this work are presented in natural units, $G = c = \hbar = k_e = 1$, and adopting the metric signature $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

2. Black holes

Black holes are solutions to the Einstein field equations. The first of such solutions was found by Kar Schwarzschild in 1916 only a year after the publication of general relativity in 1915 by Albert Einstein. It describes the gravitational field of a spherically symmetric, non-rotating and chargeless object. Other solutions, describing gravitational fields around spherically symmetric objects which are either non-rotating and charged (Reissner-Nordström black hole) or rotating and chargeless (Kerr black hole) or both, followed afterwards. We will only consider two types of black holes, the Schwarzschild and the Reissner-Norström black holes, given by the metrics

$$g_{\mu\nu} = \text{diag} \left(- \left(1 - \frac{2M}{r} \right), \left(1 - \frac{2M}{r} \right)^{-1}, r^2, r^2 \sin^2(\phi) \right), \quad (2)$$

and

$$g_{\mu\nu} = \text{diag} \left(- \left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right), \left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right)^{-1}, r^2, r^2 \sin^2(\phi) \right), \quad (3)$$

respectively. M is the mass of the black hole and Q is the charge of the black hole (in the case of the Reissner-Nordström black hole).

2.1. Coordinate singularity and tortoise coordinates

When looking at the Schwarzschild metric, one notices that at $r = 2M$ we have a singularity. This radius is called the Schwarzschild radius and it tells us that there is something wrong with our choice of coordinate system. This problem can be fixed by shifting to some other coordinate system, the tortoise coordinates, by the following transformation

$$dx = \frac{r}{r - 2M} dr, \quad (4)$$

where the line element of the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2M}{r} \right) (-dt^2 + dr^2) + r^2 d\Omega^2, \quad (5)$$

with $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. With this coordinate transformation we no longer have a singularity at $r = 2M$. The shift to tortoise coordinates makes it easy to handle the equations without worrying about singularities.

3. WKB approximation method

The motivation for the use of the WKB method is from the structure of the equation that governs the radial part of the wavefunction, which describes the perturbation of the black hole, and resembles the one dimensional Schrödinger equation for a particle with some energy scattered off a potential. The Schrödinger equation of this type is given by

$$\frac{d^2\psi}{dx^2} + k\psi = 0. \quad (6)$$

For a Schwarzschild black hole perturbed by a scalar field, the radial part of the wavefunction is describe by Eq. (6). In these calculations $k = \omega^2 - V(x)$, where ω are the quasinormal frequencies and $V(x)$ the potential (which depends on the type of black hole and perturbation). For the case of a Schwarzschild black hole perturbed by a massive scalar field, the potential is given by

$$V(x) = \left(1 - \frac{2M}{r^2}\right) \left(\mu^2 - \frac{l(l-1)}{r^2} - \frac{2M}{r^3}\right). \quad (7)$$

Detailed calculations of the QNMs for the above potential can be found in Ref. [4]. Since we can also use the WKB method solve equations of this type, we can confidently use this approximation method to solve Eq. (6) for any given black hole.

4. Boundary Conditions:

After imposing the tortoise coordinates, we will require that particles at $x \rightarrow -\infty$ (event horizon of a black hole) fall into the black hole and particles at $x \rightarrow \infty$ (spatial infinity) will no longer feel the presence of the black hole. Mathematically, this requirement translates to

$$V(x) = \begin{cases} 0, & \text{if } x \rightarrow \infty \\ 0, & \text{if } x \rightarrow -\infty. \end{cases} \quad (8)$$

The conditions on the wavefunction assume the form,

$$\Psi_{in}(x) = \begin{cases} e^{-i\omega x}, & \text{if } x \rightarrow -\infty \\ A_l e^{-i\omega x} + B_l e^{-i\omega x}, & \text{if } x \rightarrow \infty. \end{cases} \quad (9)$$

These boundary conditions are required to solve the Schrödinger-like equation for the radial part of the wavefunction.

5. Quasinormal Modes

5.1. Scalar field in Schwarzschild background

The equation of motion of a scalar field in a Schwarzschild spacetime background is given by the Klein-Gordon equation

$$\left[\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - \mu^2 \right] \Phi(x) = 0, \quad (10)$$

where $g^{\mu\nu}$ is the inverse of the Schwarzschild metric, $g_{\mu\nu}$, μ is the mass of the incoming particles and g is the determinant of the metric. Using the separation of variables

$$\Phi(x) = e^{-i\omega t} Y(\theta, \phi) R(r), \quad (11)$$

one can show (and with the use of the tortoise coordinates) that the master radial equation is given by [5]

$$\frac{d^2 R}{dx^2} + \left\{ \omega^2 - \left(1 - \frac{2M}{r^2}\right) \left(\mu^2 - \frac{l(l-1)}{r^2} - \frac{2M}{r^3}\right) \right\} R = 0. \quad (12)$$

Using the WKB approximation method, with the potential $V(r) = \left(1 - \frac{2M}{r^2}\right) \left(\mu^2 - \frac{l(l-1)}{r^2} - \frac{2M}{r^3}\right)$ we can compute numerical results for the QNFs. The table below shows some of these numerical values (generated using mathematica) for the fundamental mode ($n = 0$) and varies values for the angular quantum number l .

Table 1: Fundamental ($n = 0$) QNF ($\omega = \omega_r - i\omega_i$) for massless ($\mu = 0$) scalar field Schwarzschild black hole background:

n	l	WKB		AIM	
		Re ω	Im ω	Re ω	Im ω
0	2	0.39798	0.08826	0.5741	0.0963
	3	0.61587	0.09227		
	4	0.82212	0.09392		
1	2	0.3521	0.3213	0.5573	0.2928
	3	0.4816	0.2753		
	4	0.6794	0.3104		

In the above table we compare some WKB values that we generated and the AIM values found in [6].

5.2. Weyl field in Reissner-Nordström background

The equation of motion for the Weyl field is just the Dirac equation without the mass term

$$i\gamma^\mu \nabla_\mu \Psi(x) = 0. \quad (13)$$

The gamma matrices, γ^μ , live in curved space and are defined, in terms of vierbeins,

$$\gamma^\mu = \gamma^a e_a^\mu \quad (14)$$

with e_a^μ having one ‘‘leg’’ in flat space and the other in curved space. Vierbeins play the role of bridging flat space gamma matrices and curved space gamma matrices [7]. The vierbeins can be calculated from the condition

$$\eta_{ab} e_a^\mu e_b^\nu = g_{\mu\nu}, \quad (15)$$

The covariant derivative contains both the electromagnetic (A_μ) and the gravitational (Γ_μ) gauge fields since the Reissner-Nordström black hole is charged. Thus, we can write the covariant derivative as

$$\nabla_\mu \equiv \partial_\mu + iqA_\mu + \Gamma_\mu \quad (16)$$

with q being the coupling constant, in this case it is just the charge of the incoming Weyl particles. The Reissner-Nordström black hole background is defined by the metric,

$$g_{\mu\nu} = \text{diag} \left(\left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right), \left(1 - \frac{2M}{r} + \frac{Q}{r^2} \right)^{-1}, r^2, r^2 \sin^2(\theta) \right), \quad (17)$$

and the choice of our gamma matrices must satisfy the anticommutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (18)$$

Unlike the Schwarzschild black hole, the Reissner-Nordström black hole has two event horizons given by the form, $r_\pm = M \pm \sqrt{M^2 - Q^2}$. We can use the separation of variables again to solve

Table 2: Fundamental ($n = 0$) QNF ($\omega = \omega_r - i\omega_i$) for Weyl field in Reissner-Nordström black hole background with $\lambda = 1$ and $q = 0$:

n	WKB		
	Q	Re ω	Im ω
0	0.00	0.365718	0.193755
	0.30	0.393582	0.195632
	0.45	0.446581	0.191235

the Dirac equation, using Chandrasekhar's [8] approach, we get the radial equation to assume the form

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)Z_{\pm} = V_{\pm}Z_{\pm}, \quad (19)$$

with

$$V_{\pm} = \frac{\lambda^2\alpha}{r^2\lambda^2\sigma}(\lambda^3 \pm 1) \left[\frac{r^2\lambda^5\sigma}{\alpha^{1/2}}(1 - M) - \alpha^{1/2}\lambda^3\sigma \left(2r\lambda^2 - \frac{r^2\lambda^2}{r - r_+} + 1 \right) \right]. \quad (20)$$

Where the parameters in V_{\pm} are given by $\alpha = r^2 - 2Mr + Q^2$, $\sigma = 1 - \frac{Qq}{(r-r_+)\omega}$ and $\lambda^2 = (l + \frac{1}{2})^2$. Below we list the QNFs (computed using mathematica) for this potential.

5.3. Spin-3/2 (gravitino) field in a Reissner-Nordström background

One of the predictions of supersymmetry is the existence of supersymmetric partner for every elementary particle, called the ‘‘sparticles’’. With the postulated spin-2 graviton having a spin-3/2 (gravitino) particle. This will be the supersymmetric partner of the graviton in a supergravity theory, and has the equation of motion when free and massless as the Rarita-Schwinger equation,

$$\gamma^{\mu\nu\rho}\nabla_{\nu}\Psi_{\rho} = 0. \quad (21)$$

Similar to the two cases described above, we can make an attempt at separating the Rarita-Schwinger equation. To do this, we first have to compute the covariant derivative. This part of the paper is my current research project. Studying methods employed in [9] one can develop a similar procedure to compute QNFs (and hence the QNMs) for the spin-3/2 field in a Reissner-Nordström black hole background.

6. Conclusion

Through the separation of variables we managed to solve the Klein-Gordon equation as well as the Weyl equation in curved spacetime. Separating these equations allowed us to focus on the radial part of the wavefunction, which is Schrödinger-like, and thus compute the potential felt by the fields due to the presence of the black hole. Using the WKB approximation method, we then computed the QNFs for the scalar field (in Schwarzschild background) and the Weyl field (in the Reissner-Nordström background).

The main problem for my current project is to compute the QNFs for a spin-3/2 in a Reissner-Nordström black hole background. To build up to this calculation, in this paper we explored how

one would compute these QNFs by first considering the case for a scalar field in a Schwarzschild background and Weyl field in a Reissner-Nordström background.

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