Understanding of vector addition and subtraction by first year university students: graphical versus algebraic methods

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Abstract. The understanding of vector addition and subtraction in two-dimensions by first year university students was investigated for both the graphical representation (where vectors are represented as arrows) and the algebraic notation (using unit vectors $\widehat{i}$ and $\widehat{j}$) in a generic mathematical context. In particular, students enrolled in the 2016 first semester module General Physics for Earth Sciences at the University of Johannesburg were given tests dealing with vectors in one- and two-dimensions. Questions in these tests were structured in such a way to probe students capabilities in manipulation of vectors for different relative orientations (with aligned and/or opposing x- and y-components) for both graphical and algebraic methods. Students' performance shows that difficulties are mostly found in the use of the graphical representation, while the average performance in the algebraic format was excellent. Some of the trends include scores being comparatively higher for addition than for subtraction of vectors. For one-dimensional vectors the performance in both addition and subtraction improves when vectors point in the same direction, as opposed to when they are opposite. No such pattern is observed for two-dimensional vectors. For subtraction of vectors in one-dimension, the score is definitively worse when the vector to be subtracted points in the negative direction, because of the incapability to properly account for the negative sign given by the vector direction and the task of subtracting the same vector. Interestingly, the average performance for vector subtraction is found to be higher in two-dimensional than in one-dimensional vectors. Follow-up interviews were carried out in order to identify students' most common misconceptions.

1. Introduction

Vectors constitute a fundamental building block for any introductory physics course at university level. Vector concepts are used in topics such as motion, forces, momentum and torque, and therefore need to be properly mastered by students. Vectors are generally taught in two different representations: the graphical method (where vectors are represented as arrows in a Cartesian reference frame) and the algebraic method, where vectors are identified by means of unit vectors - $\widehat{i}$ and $\widehat{j}$ in the two-dimensional case. While the arrow representation is certainly pedagogically relevant, as it poses the accent on the directional component of a vector as a mathematical object and it helps the students to visualise physical quantities in the real space, the algebraic notation is also important as it allows for a more immediate representation of vectors in three or more dimensions, and it is useful in calculations involving vectors. Both representations are therefore necessary and complementary to each other, and it is important to understand students' perception of both.

There have been several studies dedicated to the understanding of vectors by students, and most of them have focused on the graphical representation of vectors [1, 2, 3]. Others have reported some partial comparison of the performance of students in the graphical and algebraic methods [4, 5, 6], even though the results were somewhat ambiguous [6], or different types of questions were given to different students [5], making the comparison of the results more difficult.

This study aims at gathering further insight into students' understanding of vector concepts and vector addition and subtraction in a generic mathematical context by directly comparing the students' performance in manipulating vectors in the graphical and algebraic method. This was done by means of two carefully designed tests designed to probe addition and subtraction of vectors. These tests were given to exactly the same student population, so that a one-to-one comparison of the answers could be performed. The outcome of the present investigation is in agreement with the findings reported in the literature [4, 5, 6].

2. Method

Students' understanding of vector concepts was probed by means of two tests, focusing on the addition and subtraction of vectors in one- and two-dimensions. These tests were written by 23 first year students representing the 2016 intake for the first semester course named General Physics for Earth Sciences (or PHYG01A) in the Faculty of Science at the University of Johannesburg. This is an algebra-based introductory physics course intended for students majoring in geology, geography and environmental management. The author of this paper is the lecturer of the course. It is acknowledged that the class size is rather small in terms of statistical sample, but the author's plan is to extend this investigation to larger first year physics classes at the University of Johannesburg. The tests were written by the students in class under supervision. Students were expected to work alone, and were allowed to complete the tests at their own pace.

Vectors are the second topic in the PHYG01A syllabus. They were dealt with in detail in class over a time span of 10 single lectures, which roughly takes up the first two and a half weeks of the semester. These tests were given to the students right at the end of the third week, just after the topic of vectors was completed in class. Based on the material covered during lectures, students were expected to know how to add and subtract vectors using the graphical method, as well as the algebraic method.

Both tests were set in such a way to probe the capability of adding and subtracting vectors with different relative orientations. It is important to note that the same orientation pattern was used in both tests for the graphical and the algebraic method (as it will be clarified in the following), in order to allow for a direct comparison of the performance of the students.

3. Test on one-dimensional vectors: students' performance and analysis

The test on vectors in one-dimension was composed of 32 questions, out of which 16 were meant to probe vector addition and subtraction in the graphical method, while the other 16 questions dealt with the algebraic method. Each question counted for one mark towards the overall score.

In order to obtain a more systematic understanding of student difficulties, questions probed addition and subtraction in the following four possible configurations: $\overrightarrow{A}+\overrightarrow{B}$, $\overrightarrow{A}-\overrightarrow{B}$, $-\overrightarrow{A}+\overrightarrow{B}$ and $-\overrightarrow{A}-\overrightarrow{B}$. For each configuration, combinations of vectors pointing in the same and in opposite directions were used. This resulted in four kinds of questions, as shown in figure 1 for $\overrightarrow{A}-\overrightarrow{B}$: $+(p)-(p)$, i.e. a vector in positive direction minus a vector in positive direction (a); $+(n)-(n)$, i.e. negative minus negative (b); $+(p)-(n)$, i.e. positive minus negative (c); $+(n)-(p)$, i.e. negative minus positive (d). The same orientation pattern was used for the other

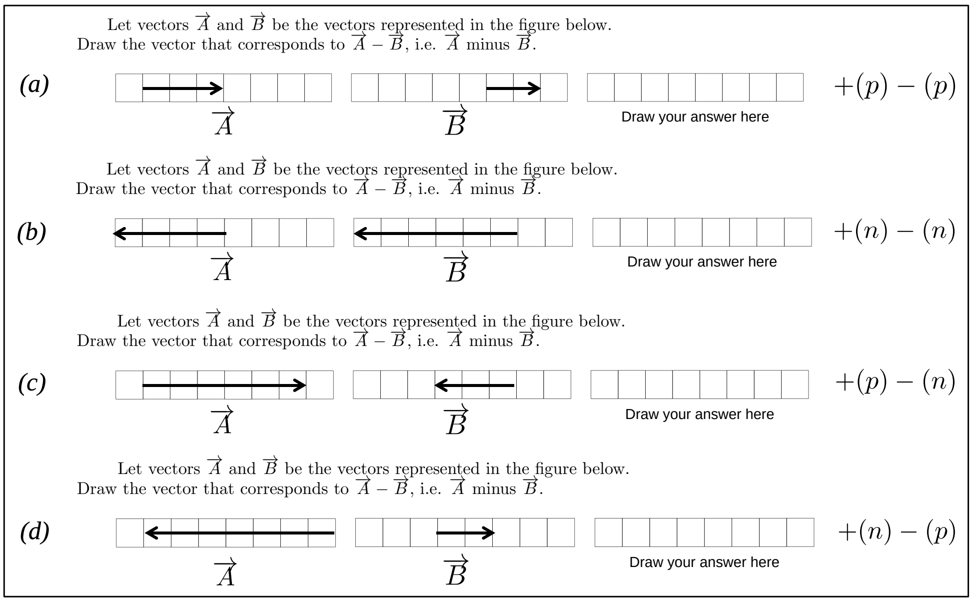

Figure 1. Examples of questions for one-dimensional vectors in the arrow format for $\vec{A} - \vec{B}$, exploiting different relative orientations between vectors $\vec{A}$ and $\vec{B}$. A one-dimensional grid was assigned to vectors $\vec{A}$ and $\vec{B}$ separately, and students were asked to draw their answer in an empty grid. Question were reiterated in symbol and word format (e.g., $\vec{A}$ *minus* $\vec{B}$) in order to better ensure that the students were not simply misreading the operation and making unnecessary mistakes.

three configurations, making the number of possible combinations to add up to 16.

Specific operations and relative orientations were also accounted for in the algebraic method, in order to obtain the one-to-one correspondence between graphical and algebraic methods, as mentioned above. The order in which questions were asked in the test was shuffled, to avoid students being able to find out the underling structure of the test.

This is an example of the algebraic method, corresponding to the graphical situation shown in figure 1 (a), i.e. $+(p) - (p)$:

Consider vectors $\vec{A} = 5\,\hat{\imath}$ and $\vec{B} = 4\,\hat{\imath}$. What is $\vec{A} - \vec{B}$? Answer:

Students' performance in this test is shown in Table 1. The comparative analysis of the results allows to extract the following trends:

(i) The performance in the algebraic method is excellent, with an overall average score of 93%, which is much higher than the performance in the graphical method. This does not depend upon whether addition or subtraction was tested.

(ii) The overall best performance in the graphical method was scored for $\vec{A} + \vec{B}$ (average of the four same-type questions = 84%), while the worst for $\vec{A} - \vec{B}$ (average = 30%). In particular the lowest score was obtained for the subtraction of vectors of the type $+(p) - (n)$. Follow-up interviews helped to clarify the reason behind this. Most of the students were confusing the negative sign given by the vector direction with the task of subtracting the same vector, and did not understand that both contributions (the negative sign and the subtraction) had to be properly accounted for. This indicates that students' difficulties associated to

Table 1. Class performance in the test on one-dimensional vectors for both graphical and algebraic methods. The difference score (graphical-algebraic) is shown in the last column.

		graphical (%)	algebraic (%)	difference (%)
$\vec{A}+\vec{B}$	$+(p)+(p)$	87	95	-8
$\vec{A}+\vec{B}$	$+(n)+(n)$	96	91	$+5$
$\vec{A}+\vec{B}$	$+(p)+(n)$	70	100	-30
$\vec{A}+\vec{B}$	$+(n)+(p)$	83	95	-12
$\vec{A}-\vec{B}$	$+(p)-(p)$	39	95	-56
$\vec{A}-\vec{B}$	$+(n)-(n)$	52	86	-34
$\vec{A}-\vec{B}$	$+(p)-(n)$	13	95	-82
$\vec{A}-\vec{B}$	$+(n)-(p)$	17	86	-69
$-\vec{A}+\vec{B}$	$-(p)+(p)$	87	100	-13
$-\vec{A}+\vec{B}$	$-(n)+(n)$	48	90	-42
$-\vec{A}+\vec{B}$	$-(p)+(n)$	13	76	-63
$-\vec{A}+\vec{B}$	$-(n)+(p)$	30	90	-60
$-\vec{A}-\vec{B}$	$-(p)-(p)$	61	100	-39
$-\vec{A}-\vec{B}$	$-(n)-(n)$	48	95	-47
$-\vec{A}-\vec{B}$	$-(p)-(n)$	43	100	-57
$-\vec{A}-\vec{B}$	$-(n)-(p)$	22	90	-68

the presence of a negative sign depend on the direction of the vectors as well. Surprisingly in fact the average score for $-\vec{A}+\vec{B}$ and $-\vec{A}-\vec{B}$ was 44%, definitively higher than for $\vec{A}-\vec{B}$.

(iii) Focusing on the relative orientation between vectors in the graphical method, scores were definitively higher when vectors point in the same direction, than when they are opposite. This is consistent across the four different configurations. Interviews revealed that the most common misconception is that opposing arrows are always "acting against each other" (and it does not matter which mathematical operation, addition or subtract, is asked for), and the task is to find which one "wins" in this competition. This is effectively equivalent to always adding vectors, which obviously leads to mistakes.

4. Test on two-dimensional vectors: students' performance and analysis

The test on two-dimensional vectors was again composed of 32 questions, evenly split between the graphical method and the algebraic notation, keeping the usual one-to-one correspondence between the two. Each question counted for one mark towards the overall score. In this second test only addition ($\vec{A}+\vec{B}$) and subtraction ($\vec{A}-\vec{B}$) were probed, for situations where the two vectors were either aligned in both dimensions, or where they were aligned in one dimension but opposed in the other. The aim was to probe whether the same pattern of mistakes encountered by students in the one-dimensional case persisted in two-dimensions. If this was the case, then we would expect questions with vectors having one opposite component to score significantly less than when both components are aligned.

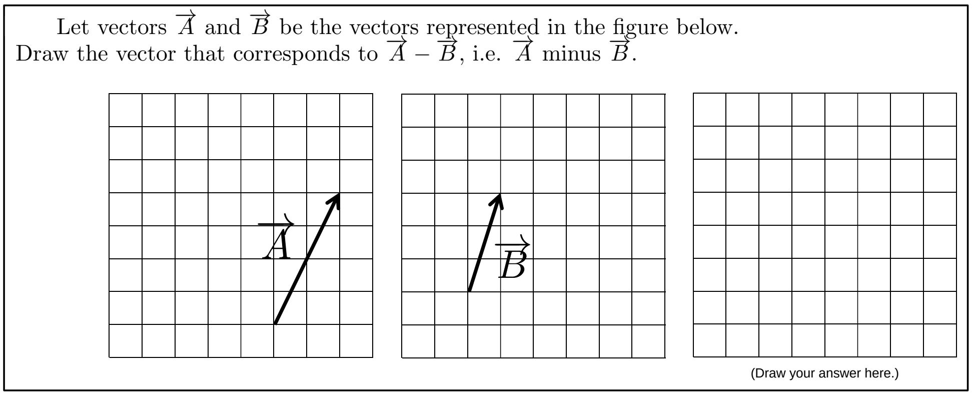

Figure 2. Example of two-dimensional question in the arrow format.

Table 2. Class performance in the test on two-dimensional vectors for both graphical and algebraic methods. The difference score (graphical-algebraic) is shown in the last column. "+" and "−" indicate the orientation of vector components A_x, A_y, B_x and B_y.

	A_x	A_y	B_x	B_y	graphical (%)	algebraic (%)	difference (%)
$\vec{A}+\vec{B}$	+	+	+	+	59	91	−32
$\vec{A}+\vec{B}$	−	−	−	−	59	91	−32
$\vec{A}+\vec{B}$	+	−	+	−	45	95	−50
$\vec{A}+\vec{B}$	−	+	−	+	50	100	−50
$\vec{A}-\vec{B}$	+	+	+	+	41	95	−54
$\vec{A}-\vec{B}$	−	−	−	−	32	95	−63
$\vec{A}-\vec{B}$	+	−	+	−	59	86	−27
$\vec{A}-\vec{B}$	−	+	−	+	55	95	−40
$\vec{A}+\vec{B}$	+	+	+	−	59	95	−36
$\vec{A}+\vec{B}$	−	−	+	−	73	100	−27
$\vec{A}+\vec{B}$	−	+	+	+	73	91	−18
$\vec{A}+\vec{B}$	−	+	−	−	50	95	−45
$\vec{A}-\vec{B}$	+	+	+	−	59	90	−31
$\vec{A}-\vec{B}$	−	−	+	−	55	95	−40
$\vec{A}-\vec{B}$	−	+	+	+	42	86	−41
$\vec{A}-\vec{B}$	−	+	−	−	32	86	−54

Figure 2 shows an example of how a two-dimensional question in the arrow format was asked. Vectors $\vec{A}$ and $\vec{B}$ were drawn in two different two-dimensional grids, and the students were expected to draw the final answer in a third grid. For the particular example reported in this figure both x- and y-components of the vectors are positive (please note that students were informed of this convention). The equivalent question asked in the algebraic format was as follows:

$\boxed{\text{Consider vectors } \overrightarrow{A} = 2\,\widehat{\imath} + 3\,\widehat{\jmath} \text{ and } \overrightarrow{B} = 3\,\widehat{\imath} + 8\,\widehat{\jmath}\text{. What is } \overrightarrow{A} - \overrightarrow{B}\text{? Answer:}}$

Students' performance in this second test is shown in table 2. From these results the following considerations can be drawn:

(i) The score for the algebraic method is very high, with an average score of 93% over the 16 questions that were asked; this is the same as the average score in the one-dimensional case.

(ii) As far as the graphical method is concerned, scores were lower than the algebraic one, and in particular slightly higher for addition than for subtraction; interestingly, this does not depend upon the relative orientation of the components. In short, students did not seem to perform better when vectors were aligned as in the one-dimensional case.

(iii) Again for the graphical method, the direct comparison of the scores in one- and two-dimensions for vector subtraction $\overrightarrow{A} - \overrightarrow{B}$ specifically, shows that surprisingly students scored higher in two-dimensions. The average score in two-dimensions was in fact 47% against a 30% scored in one-dimension. Again, post-test interviews were found to be a very important tool in order to understand this finding. The scenario that emerged is the following: students have a very strong perception of the direction for one-dimensional vectors (in other words a vector is perceived as being either positive or negative); this leads to them facing the task of subtracting vectors with some kind of "preconception" (for example that opposing arrows are always in competition) on what the outcome should be. This spacial awareness is much weaker in two dimensions; students are then more free from preconceptions and able to apply the procedure for vector subtraction more successfully.

5. Concluding remarks

In conclusion, first year students' understanding of vector concepts was investigated for both the graphical and the algebraic methods of vector representation. Students were given two tests to probe their capability to add and subtract one- and two-dimensional vectors with different relative orientations. In general, the score in the algebraic method type questions was averaged above 90%, while the performance in the graphical method was far less successful. As far as the graphical method is concerned, students generally performed better in the addition than in the subtraction of vectors. But while the score for one-dimensional vectors is higher if vectors are aligned, there is no such trend for two-dimensional vectors. Having to deal with aligned or opposite vector components does not clearly impact on the correctness of the final answer.

Interestingly, the scores for subtraction of vectors in the graphical method were higher in two-dimensions as opposed to the one-dimensional case. From post-test interviews this can be understood by considering the weaker perception of space and vector direction that students have for the two-dimensional case. Giving a post-test to the same students, which should be structured in a similar way to the first one and should follow one-to-one and class consultation on the issues raised from the first test, would help understand how addressing such issues explicitly would improve class performance. An interestingly follow-up of this study could then be the investigation of the relationship between students' perception of space and performance in vector graphical manipulation, and how the two affect each other in both one- and two-dimensions.

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