Cosmological Chaplygin gas as modified gravity

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Abstract. In the scramble for the understanding of the nature of dark matter and dark energy, it has recently been suggested that the change of behavior of the missing energy density might be regulated by the change in the equation of state of the background fluid. The Chaplygin Gas (CG) model in cosmology is one of the most profound candidates for this suggestion. This work aims to bring to light a geometric interpretation of the model by re-writing the different toy models in terms of exact $f(R)$ gravity solutions that are generally quadratic in the Ricci scalar with appropriate ΛCDM limiting solutions.

1. Introduction

The Chaplygin gas model in FLRW background provides a cosmic expansion history with a universe filled with an exotic background fluid. The model consists of a universe that transits from a decelerating matter-dominated phase to a late-time accelerated one. However, in its intermediate stages, it behaves as a mixture of a cosmological constant and a perfect fluid obeying the $p = w\rho$ equation of state. One can, therefore, mimic the background evolution history of the Universe, including the current observation of cosmic acceleration, using such models [1, 2, 3, 4, 5, 6].

The CG model was introduced first by Chaplygin [7] as a model for aerodynamical studies. In its original form, the mode has an equation of state in the form

$$p = -\frac{A}{\rho^\alpha} \; , \tag{1}$$

where p and ρ are respectively pressure and energy density in a comoving reference frame with $\rho > 0$, and A and α are positive constants. Chaplygin considered the case with $\alpha = 1$, but a more *generalised Chaplygin gas* (GCG) equation of state is obtained when $0 \leq \alpha \leq 1$ [8, 9, 10]. Under homogeneity considerations, the relativistic energy conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{2}$$

in the context of FRW cosmology substituting for the equation of state yields an expression for the density in terms of the scale factor $a(t)$ given as

$$\rho = \left(A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} \; , \tag{3}$$

where B here is a positive integration constant. For small values of a ($a^6 \ll B/A$) the energy density is approximated by

$$\rho \sim \frac{\sqrt{B}}{a^3} \,, \tag{4}$$

which clearly corresponds to a dust-like dominated phase. For large values of the cosmological radius a, it follows that

$$p \simeq -\rho \Rightarrow \quad \rho \sim \sqrt{A} \Rightarrow \quad p \sim -\sqrt{A} \,, \tag{5}$$

which corresponds to an empty universe with a cosmological constant $\sqrt{A}$ and that is a de Sitter universe, through an intermediate regime described by the equation of state for stiff matter $p = \rho$. There is a possibility of interpreting the model as a "quintessential" model [6, 2, 11] with a critical density

$$\rho_c = (A + B)^{\frac{1}{1+\alpha}} \,, \tag{6}$$

whereas the Hubble parameter is given through the Friedmann equation by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho = \frac{1}{3}\left(A + \frac{B}{a^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}} \,. \tag{7}$$

This model automatically leads to an asymptotic phase where the equation of state is dominated by a cosmological constant $\sqrt{A}$. Subsequently, it has been shown that this model admits, under appropriate conditions, an inhomogeneous generalisation, which can be regarded as a unification of dark matter and dark energy models [1]. We can express the perturbed pressure as

$$\delta p = \frac{A}{\rho^2}\delta\rho \,, \tag{8}$$

and for the density contrast we have

$$\delta = \frac{\delta\rho}{\rho} \propto t^{2/3} \,. \tag{9}$$

A very important point here is that the evolution of density perturbations in a universe dominated by the Chaplygin gas admits an initial phase of growing perturbations, with the same rate as in the dust case of the cosmological standard mode [12]. Furthermore, the model predicts an increasing value for the effective cosmological constant [2], *i.e*, in the context of a Chaplygin cosmology, once an expanding universe starts accelerating it cannot decelerate any more, a fact that we seem to be observing today.

For open or flat Chaplygin cosmologies ($k = -1, 0$), the Universe always evolves from a decelerating to an accelerating epoch. For the closed Chaplygin cosmological models ($k = 1$), Einstein static universes have

$$B = \frac{2}{3\sqrt{3A}} \,. \tag{10}$$

The GCG cosmological model, with no additional fluid components, is compatible with structure formation and large-scale structure only for α sufficiently small ($\alpha < 10^{-5}$), in which case it is indistinguishable from the ΛCDM model.

The GCG of Eq. (1) is important to cosmology and it is one of the promising candidates to explain the present accelerated expansion of the universe with simple model unifying dark matter and dark energy [1, 12, 13] as manifestations of a single cosmic fluid [14, 15, 16, 17].

The *modified Chaplygin gas* (MCG) model is often used to describe the acceleration phase of the Universe from the radiation era to the ΛCDM model. It includes a matter term [8, 18]

$$p = \gamma\rho - \frac{A}{\rho^\alpha} , \quad 0 \le \alpha \le 1 , \tag{11}$$

where $0 < \gamma < 1/3$, $0 < \alpha < 1$ and B is a positive constant. It has been shown that $A = 1/3$ is the best fitted value to describe the evolution of the Universe from radiation regime to the Λ-cold dark matter regime [19]. The energy density for such a model is given by

$$\rho = \left[\frac{A}{1+\gamma} + \frac{C}{a^{3(1+\gamma)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} , \tag{12}$$

where C is an arbitrary integration constant. This model is a more appropriate choice to have constant negative pressure at low energy density and high pressure at high energy density. We can use this equation of state to describe low-surface brightness galaxies which are supposed to be dominated by dark matter [20]. The MCG equation of state is a suitable description for an ordinary linear barotropic fluid, but there are other barotropic fluids with equation of state being quadratic and higher orders. For example, recently the model has been extended [21, 22, 23, 24] so that the resulting equation of state can also recover a barotropic fluids with higher orders

$$p_c = \sum_{i=1}^{n} \gamma_i \rho_c^i - \frac{A}{\rho^\alpha} , \tag{13}$$

where p_c and ρ_c are the pressure and energy density of the extended Chaplygin gas which is the unification of the dark matter and dark energy. There is no general solution to this approach. But if we reduce $n = 1$ the above expression recovers the standard MCG. Barotropic fluids with quadratic equation of state can be recovered by setting $n = 2$, reducing Eq. (13) to

$$p_c = \gamma_1 \rho_c + \gamma_2 \rho_c^2 - \frac{A}{\rho_c^\alpha} , \tag{14}$$

where γ_1, γ_2, A and α are positive constants. These models give us the second-order solution

$$\rho_c = \left[\frac{A}{1+\gamma_1} + \frac{C}{a^{3(1+\gamma_1)(1+\alpha)}} e^{-(1+\alpha)(1+\gamma_1)f(\rho_c)} \right]^{\frac{1}{1+\alpha}} , \tag{15}$$

where $C = (1+\gamma_1)^{-1}$ and

$$f(\rho_c) = \frac{\gamma_2 \rho_c}{(1+\gamma_1)^2} - \frac{A\gamma_2 \rho_c}{(1+\alpha)(1+\gamma_1)^2((1+\gamma_1)\rho_c^{1+\alpha} - A)}$$
$$+ \gamma_2 \int \frac{A(2+\alpha)}{(1+\alpha)(1+\gamma_1)^2((1+\gamma_1)\rho_c^{1+\alpha} - A)} d\rho_c . \tag{16}$$

Note that in case of $\gamma_2 = 0$, we have a vanishing $f(\rho_c)$. With higher order n, we will recover a higher-order barotropic fluid.

The *generalized cosmic Chaplygin gas* (GCCG) models are those Chaplygin gas models that admit the equation of state given by [25]

$$p = -\rho^{-\alpha} \left[C + (\rho^{1+\alpha} - C)^{-\omega} \right] , \tag{17}$$

where

$$C = \frac{\gamma}{1 + \omega} - 1 \, , \tag{18}$$

with γ a constant which now can take on both positive and negative values, and $0 > \omega > -l$, l being a positive definite constant which can take on values larger than unity. In the special case when $\omega = 0$ one can write that $C = \gamma - 1$. The speciality of this model is its stability so the theory is free from unphysical behaviours even when the vacuum fluid satisfies the phantom energy condition [19]. The above equation of state satisfies the following conditions:

- it becomes a de Sitter fluid at late time and when $\omega = -1$,
- it reduces to $p = \omega\rho$ in the limit that the Chaplygin parameter $\gamma \to 0$,
- it reduces to the equation of state of current Chaplygin unified dark matter models at high energy density,
- the evolution of density perturbations becomes free from any pathological behaviour of the matter power spectrum for physically reasonable values of the involved parameters at late times.

By integrating the continuity equation (2) we get for the energy density

$$\rho(a) = \left[C + \left(1 + \frac{A}{a^{3(1+\alpha)(1+\omega)}} \right)^{\frac{1}{1+\omega}} \right]^{\frac{1}{1+\alpha}} , \tag{19}$$

where B is a positive integration constant. B shows the effect of Chaplygin gas, and the cosmic effect represented by ω. A further extension of the CG model is called *modified cosmic Chaplygin gas* (MCCG) [26, 19], where the EOS is further generalized to

$$p = \gamma\rho - \frac{1}{\rho^\alpha} \left[\frac{A}{1 + \omega} - 1 + \left(\rho^{1+\alpha} - \frac{A}{1 + \omega} + 1 \right)^{-\omega} \right] . \tag{20}$$

Here A and γ could be both positive or negative constants, and $-l < \omega < 0$ where l is a positive definite constant with values larger than unity. Here also, $0 < \alpha \leq 1$, and the case where $\omega = 0$ gives the equation of state corresponding to the MCG. If we then put $\gamma = 0$, the equation of state corresponding to the GCG is recovered. We can also reach back to the simplest case where $\alpha = 1$, the Chaplygin gas's original equation of state.

Chaplygin gas models have been studied in flat Friedmann models, in terms of the recently proposed "statefinder" parameters [27], dimensionless parameters that allow us to characterise the properties of dark energy in a model-independent manner. It has also been shown that the simple flat Friedmann model with Chaplygin gas can equivalently be described in terms of a homogeneous minimally coupled scalar field ϕ, which has been used in a variety of inflationary models in describing the transition from the quasi-exponential expansion of the early universe to a power law expansion in order to understand the present acceleration of the Universe [2, 28].

The model can be re-expressed as flat Friedman universes containing a scalar field with particular self-interaction potentials [29, 30]; in other words, a very light scalar field ϕ whose effective potential $V(\phi)$ leads to an accelerated phase at the late stages of the Universe [31] by constructing models where the matter responsible for such behaviour is also represented by a scalar field [32, 33].

In [2, 34] a homogeneous scalar field $\phi(t)$ and a potential $V(\phi)$ have been shown to describe Chaplygin cosmology. An extended work is done by [35] with a modified CG. Moreover, the Chaplygin gas is the only gas known to admit a supersymmetric generalisation [36].

2. Chaplygin gas as $f(R)$ gravity?

The main objective of this work is to study models of $f(R)$ gravity which, when we impose the Chaplygin gas equations of state (EoS) to their effective pressure and energy density, produce viable exact solutions that reduce to the ΛCDM scenario in the approximate cosmological limits [37], thus giving a geometric interpretation of Chaplygin gas cosmological models. But first let us briefly discuss $f(R)$ models of gravity. These are models of modified gravity introduced to address the problems and shortcomings of GR, especially in light of the current accelerated epoch of cosmic expansion. They involve modified Einstein-Hilbert action of the form

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} \left[f(R) + 2\mathcal{L}_m \right] . \tag{21}$$

The generalised field equations, using the standard variational principle w.r.t the metric, are given by

$$f'G_{ab} = T^m_{ab} + \frac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f' , \tag{22}$$

where f', etc. are shorthands derivatives of $f = f(R)$ w.r.t the Ricci scalar R, and T^m_{ab} is the energy-monetum tensor for matter.

These models are among the most widely studied modified theories of gravity [38, 39, 40, 41]. The extra degree of freedom in these models gives us the freedom to explain accelerated cosmic expansion and structure formation of the Universe without adding any extra form of exotic matter.

Assuming the Universe is filled with standard matter and curvature sources, its total energy density, isotropic pressure, anisotropic pressure and heat flux terms are given, respectively, by

$$\rho \equiv \frac{\rho_m}{f'} + \rho_R , \quad p \equiv \frac{p_m}{f'} + p_R , \quad \pi_{ab} \equiv \frac{\pi^m_{ab}}{f'} + \pi^R_{ab} , \quad q_a \equiv \frac{q^m_a}{f'} + q^R_a . \tag{23}$$

Up to first-order perturbations, the curvature fluid components are defined as:

$$\rho_R = \frac{1}{f'} \left[\tfrac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R \right] , \tag{24}$$

$$p_R = \frac{1}{f'} \left[\tfrac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \tfrac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R \right) \right] , \tag{25}$$

$$q^R_a = -\frac{1}{f'} \left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \tfrac{1}{3}f''\Theta\tilde{\nabla}_a R \right] , \tag{26}$$

$$\pi^R_{ab} = \frac{f''}{f'} \left[\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R} \right] . \tag{27}$$

For FLRW spacetimes, the Ricci scalar R is given by

$$R = 2\dot{\Theta} + \frac{4}{3}\Theta^2 , \tag{28}$$

where Θ is the cosmic expansion parameter related to the cosmological scale factor $a(t)$ and the Hubble parameter $H(t)$ via the equations

$$\Theta \equiv 3\frac{\dot{a}(t)}{a(t)} = 3H(t) . \tag{29}$$

If the Ricci scalar varies slowly, i.e., if R is almost constant in time, during steady-state exponential expansion in a de Sitter spacetime (such as during inflation or late-time evolution), the approximation $\dot{\Theta} \to 0$ results in

$$R = \frac{4}{3}\Theta^2 = \text{const} . \tag{30}$$

Using this approximation, if we consider the background curvature energy density and isotropic pressure terms defined in Eqs. (24) and (25) above, we obtain the simple relation

$$\rho_R = \frac{1}{4}\left[R(f'+1) - 2f\right] = -p_R \ . \tag{31}$$

This equation of state, with an effective EoS parameter $w_R = -1$, provides the condition for an exponential (accelerated) expansion with a constant Hubble parameter. The energy density ρ_R (with its negative pressure p_R) remains constant and can be interpreted as playing the role of the cosmological constant Λ.

The plan here is, given the different equations of state for the Chaplygin gas models, to come up with models of $f(R)$ gravity that can mimic those models. In other words, we want to do away with these exotic matter forms (Chaplygin gas models) in favor of a modification in the gravitational Lagrangian, i.e., geometry. To solve for the appropriate forms of $f(R)$ corresponding to different Chaplygin gas equations of state, we solve the resulting differential equations using the symbolic computational packages of *Maple*. This is possible because of the nature of the approximations made, namely, the slowly-changing Ricci scalar assumption. Such (analytic) solutions are intractable if this assumption is relaxed, and a future work will involve a fully numerical computation of such solutions, once the appropriate initial conditions are fully understood.

Comparing the original and generalized Chaplygin gas EoS (1) with (31), we obtain

$$p_R = -\rho_R = -\frac{A}{\rho_R^\alpha} \ , \tag{32}$$

which, upon using Eq. (24), leads to the o.d.e

$$R\frac{f(R)}{dR} - 2f(R) + R = 4A^{\frac{1}{\alpha+1}} \ . \tag{33}$$

Solving this o.d.e yields

$$f(R) = R + C_1 R^2 - 2A^{\frac{1}{\alpha+1}} \tag{34}$$

for an arbitrary (integration) constant C_1. We note that the ΛCDM solution $f(R) = R - 2\Lambda$ is already a particular solution with $C_1 = 0$ and $A = \Lambda^{\alpha+1}$. In particular, if $\alpha = 0$, then $A = \Lambda$, from which, going back to Eq. (32), one concludes $\rho_R = \Lambda$.

If we include the linearized Laplacian term in Eqs. (24) and (25) and use the eigenvalue $-\frac{k^2}{a^2}$ of the covariantly defined Laplace-Beltrami operator $\tilde{\nabla}^2$ on (almost) FLRW spacetimes

$$\tilde{\nabla}^2 R = -\frac{k^2}{a^2}R \tag{35}$$

for a comoving wavenumber k, we obtain the second-order o.d.e

$$D^2 R f''(R) - R f'(R) + 2f(R) - R + 4A^{\frac{1}{\alpha+1}} = 0 \ , \tag{36}$$

where here we have defined

$$D^2 \equiv \frac{4(2+3\alpha)}{3(1+\alpha)}\frac{k^2}{a^2} \ . \tag{37}$$

The solution of Eq. (36) is given, for arbitrary constants C_2, C_3, by

$$f(R) = R + C_2\left[R^2 - 2RD^2\right] + C_3\left[(R^2 - 2RD^2)Ei\left(1, -\frac{R}{D^2}\right) + (R - D^2)D^2 e^{\frac{R}{D^2}}\right] - 2A^{\frac{1}{\alpha+1}} \ , \tag{38}$$

which should reduce to the quadratic solution (34) for negligible values of D^2, i.e., for small first-order contributions to the energy density and pressure terms.

For the MCG EoS of Eq. (11), we can write

$$p_R = \gamma \rho_R - \frac{A}{\rho_R^\alpha} \, , \tag{39}$$

and the resulting $f(R)$ model generalizes to

$$f(R) = R + C_4 R^2 - 2 \left(\frac{A}{\gamma + 1} \right)^{\frac{1}{\alpha+1}} \, , \tag{40}$$

where C_4 is an arbitrary integration constant.

The ΛCDM solution is a limiting case of this generalized model when $C_4 = 0$ and $A = (\gamma + 1)\Lambda^{\alpha+1}$. In particular, if $\alpha = 0 = \gamma$, then $A = \Lambda$.

Following similar arguments as in the preceding subsection, if we include the linearized Laplacian contributions to the energy density and pressure, we get Eq. (36) generalized to

$$D^2 R f''(R) - R f'(R) + 2 f(R) - R + 4 \left(\frac{A}{\gamma + 1} \right)^{\frac{1}{\alpha+1}} = 0 \, , \tag{41}$$

the solution of which can be given by

$$f(R) = R + C_5 \left[R^2 - 2RD^2 \right] + C_6 \left[(R^2 - 2RD^2)Ei\left(1, -\frac{R}{D^2} \right) + (R - D^2)D^2 e^{\frac{R}{D^2}} \right]$$
$$- 2 \left(\frac{A}{\gamma + 1} \right)^{\frac{1}{\alpha+1}} \, , \tag{42}$$

for an arbitrary integration constants C_5 and C_6. This solution obviously generalizes Solutions (34),(38) and (40) and should reduce to the quadratic solution (34) for vanishingly small B^2 values. In [42], it has been shown that any quadratic Lagrangian leading to an isotropic, homogeneous cosmological model takes the form

$$f(R) = R - 2\Lambda - \frac{1}{6}\beta R^2 \, , \tag{43}$$

where β is an arbitrary, real constant. If we keep only the quadratic solution in (42), i.e., if we set $C_6 = 0$, the Lagrangian (43) corresponds to the choice

$$C_5 = -\frac{1}{6}\beta \, , D = 0 \, , A = (\gamma + 1)\Lambda^{\alpha+1} \, . \tag{44}$$

Another interesting fact worth pointing out here is that the condition for the existence of a maximally symmetric vacuum solution in $f(R)$ gravity [42]

$$R_0 f'(R_0) = 2f(R_0) \tag{45}$$

leads to the quadratic solution resulting in the constraint

$$R_0 \left(1 - 2C_5 D^2 \right) - 4 \left(\frac{A}{\gamma + 1} \right)^{\frac{1}{\alpha+1}} = 0 \, . \tag{46}$$

The corresponding GR de Sitter, anti-de Sitter and Minkowski solutions $R_0 = 4\Lambda$ (respectively for $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$) are obtained when $C_5 D^2 = 0$ and $A = (\gamma + 1)\Lambda^{\alpha+1}$.

The so-called *modified generalized Chaplygin gas* (mGCG) model is described by a barotropic equation of state of the form [43, 44]

$$p = \beta\rho - (1 + \beta)\frac{A}{\rho^\alpha} \ . \tag{47}$$

Models of $f(R)$ gravity that satisfy the condition (31), at the sam time mimicking the mGCG, can be shown to be governed by the same equation as (33) and admit the same solutions (34), provided $\beta \neq -1$. On the other hand, if linearized Laplacian terms are included, then the corresponding differential equation in $f(R)$ generalizes to

$$E^2 R f''(R) - R f'(R) + 2f(R) - R + 4A^{\frac{1}{\alpha+1}} = 0 \ , \tag{48}$$

where we have defined

$$E^2 \equiv \frac{4\left[2 + 3\alpha + 3\beta(1 + \alpha)\right]}{3(1 + \alpha)(1 + \beta)}\frac{k^2}{a^2} \ . \tag{49}$$

Worthy of note is that this equation and its solution

$$f(R) = R + C_7\left[R^2 - 2RE^2\right] + C_8\left[(R^2 - 2RE^2)Ei\left(1, -\frac{R}{E^2}\right) + (R - E^2)E^2 e^{\frac{R}{E^2}}\right]$$
$$- 2\left(\frac{A}{\gamma + 1}\right)^{\frac{1}{\alpha+1}} \ , \tag{50}$$

reduce to their *generalized* counterparts of Eqs. (36) and (38) when $\beta = 0$.

3. Conclusion

To summarize, we have explored exact $f(R)$ gravity solutions that mimic Chaplygin-gas inspired ΛCDM cosmology for the so-called original, generalized, modified and generalized modified Chaplygin gas equations of state. The resulting solutions are generally quadratic in the Ricci scalar, but have appropriate ΛCDM solutions as their limiting cases. These solutions, given appropriate initial conditions, can be potential candidates for scalar field-driven early universe expansion (inflation) and dark energy-driven late-time cosmic acceleration.

The solutions discussed here are based on a slowly-changing Ricci curvature assumption and can generally be obtained using symbolic computing environments such as *Maple*. However, more realistic solutions should relax this assumption, and consider higher-order corrections as well. Such models are generally complex to solve, and advanced numerical computations require more physically motivated initial conditions, currently not fully understood, but which the authors would like to explore further in a future work.

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