Nonlinear dynamics of laser systems with elements of a chaos: Advanced computational code

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Abstract. A general, uniform chaos-geometric computational approach to analysis, modelling and prediction of the non-linear dynamics of quantum and laser systems (laser and quantum generators system etc) with elements of the deterministic chaos is briefly presented. The approach is based on using the advanced generalized techniques such as the wavelet analysis, multi-fractal formalism, mutual information approach, correlation integral analysis, false nearest neighbour algorithm, the Lyapunov's exponents analysis, and surrogate data method, prediction models etc There are firstly presented the numerical data on the topological and dynamical invariants (in particular, the correlation, embedding, Kaplan-York dimensions, the Lyapunov's exponents, Kolmogorov's entropy and other parameters) for laser system (the semiconductor GaAs/GaAlAs laser with a retarded feedback) dynamics in a chaotic and hyperchaotic regimes.

1. Introduction

In a modern computational quantum and laser physics, electronics and others there are studied various systems and devices (such as atomic and molecular systems in an electromagnetic field, multi-element semiconductors and gas lasers etc), dynamics of which can exhibit a chaotic behaviour. These systems can be considered in the first approximation as a grid of autogenerators (quantum generators), coupled by different way [1-10]. It is easily to understand that a quantitative studying of the chaos phenomenon features is of a great interest and importance for many scientific and technical applications. At the present time it became one of the most actual and important problems of computational physics of the complex non-linear systems.

In this work we firstly applied a general, uniform chaos-geometric formalism to analysis and modelling of non-linear dynamics of the laser systems with elements of a chaos. The formalism is based on using the advanced generalized techniques such as the wavelet analysis, multi-fractal formalism, mutual information approach, correlation integral analysis, false nearest neighbour algorithm, the Lyapunov's exponents analysis, and surrogate data method, prediction models etc (see details in Refs. [6-19]). There are firstly presented the numerical data on topological and dynamical invariants of chaotic systems, in particular, the correlation, embedding, Kaplan-York dimensions, the Lyapunov's exponents, Kolmogorov's entropy etc for laser (the semiconductor GaAs/GaAlAs laser with retarded feedback) systems dynamics in chaotic and hyperchaotic regimes.

2. Chaos–geometric approach to modelling chaotic dynamics of the laser systems: General formalism

The important step of the quantitative studying chaotic dynamics of different dynamical systems is an numerical analysis of the characteristic time series, i.e. the time series of the key dynamical characteristics. Let us formally consider scalar measurements $s(n) = s(t_0 + n\Delta t) = s(n)$, where t_0 is the start time, Δt is the time step, and is n the number of the measurements. In a general case, $s(n)$ is any time series, particularly the amplitude level. Since processes resulting in the chaotic behaviour are fundamentally multivariate (look [18-23]), it is necessary to reconstruct phase space using as well as possible information contained in the $s(n)$. Such a reconstruction results in a certain set of d-dimensional vectors $\mathbf{y}(n)$ replacing the scalar measurements. Packard et al. [21] introduced the method of using time-delay coordinates to reconstruct the phase space of an observed dynamical system. The direct use of the lagged variables $s(n + \tau)$, where τ is some integer to be determined, results in a coordinate system in which the structure of orbits in phase space can be captured. Then using a collection of time lags to create a vector in d dimensions,

$$\mathbf{y}(n) = [s(n), s(n + \tau), s(n + 2\tau), \ldots, s(n + (d-1)\tau)], \quad (1)$$

the required coordinates are provided. In a nonlinear system, the $s(n + j\tau)$ are some unknown nonlinear combination of the actual physical variables that comprise the source of the measurements. The dimension d is called the embedding dimension, d_E.

According to Mañé and Takens [22,23], any time lag will be acceptable is not terribly useful for extracting physics from data. If τ is chosen too small, then the coordinates $s(n + j\tau)$ and $s(n + (j + 1)\tau)$ are so close to each other in numerical value that they cannot be distinguished from each other. Similarly, if τ is too large, then $s(n + j\tau)$ and $s(n + (j + 1)\tau)$ are completely independent of each other in a statistical sense. Also, if τ is too small or too large, then the correlation dimension of attractor can be under- or overestimated respectively. It is therefore necessary to choose some intermediate (and more appropriate) position between above cases. First approach is based on computing the linear autocorrelation function. Another approach is based on using the average mutual information.

The next principal step is to determine an embedding dimension determination and to reconstruct a Euclidean space R^d large enough so that the set of points d_A can be unfolded without ambiguity. In accordance with the embedding theorem, the embedding dimension, d_E, must be greater, or at least equal, than a dimension of attractor, d_A, i.e. $d_E > d_A$. In other words, we can choose a fortiori large dimension d_E, e.g. 10 or 15, since the previous analysis provides us prospects that the dynamics of our system is probably chaotic. However, two problems arise with working in dimensions larger than really required by the data and time-delay embedding [24]. Firstly, many of computations for extracting interesting properties from the data require search and other operations in R^d whose computational cost rises exponentially with d. Secondly, but more significant from the physical viewpoint, in the presence of noise or other high dimensional contamination of the observations, the extra dimensions are not populated by dynamics, already captured by a smaller dimension, but entirely by the contaminating signal. In too large an embedding space one is unnecessarily spending time working around aspects of a bad representation of the observations which are solely filled with noise. Further it is necessary to determine the dimension d_A. There are a few standard approaches to reconstruct an attractor dimension (see, e.g., [1-8]), but usually there are applied only two methods. The first correlation integral analysis uses the correlation integral, $C(r)$, to distinguish between chaotic and stochastic systems. To compute the correlation integral, the algorithm of Grassberger and Procaccia [25] is the most commonly used approach, where the correlation integral is

$$C(r) = \lim_{N\to\infty} \frac{2}{N(n-1)} \sum_{\substack{i,j \\ (1\le i<j\le N)}} H\left(r - \| \mathbf{y}_i - \mathbf{y}_j \|\right) \quad (2)$$

where H is the Heaviside step function with $H(u) = 1$ for $u > 0$ and $H(u) = 0$ for $u \le 0$, r is the radius of sphere centered on $\mathbf{y}_i$ or $\mathbf{y}_j$, and N is the number of data measurements. If the time series is characterized by an attractor, then the integral $C(r)$ is related to the radius r given by

$$d = \lim_{\substack{r \to 0 \\ N \to \infty}} \frac{\log C(r)}{\log r}, \tag{3}$$

where d is correlation exponent that can be determined as the slop of line in the coordinates log $C(r)$ versus log r by a least-squares fit of a straight line over a certain range of r, called the scaling region.

If the correlation exponent attains saturation with an increase in the embedding dimension, then the system exhibits a chaotic dynamics. The saturation value of the correlation exponent is defined as the correlation dimension (d_2) of an attractor. The nearest integer above the saturation value provides the minimum or optimal embedding dimension for reconstructing the phase-space or the number of variables necessary to model the dynamics of the system.

To verify the results obtained by the correlation integral analysis, one can use surrogate data method. This method (look [6,7,26-29]) is an approach that makes use of the substitute data generated in accordance to the probabilistic structure underlying the original data. This means that the surrogate data possess some of the properties, such as the mean, the standard deviation, the cumulative distribution function, the power spectrum, etc., but are otherwise postulated as random, generated according to a specific null hypothesis. Here, the null hypothesis consists of a candidate linear process, and the goal is to reject the hypothesis that the original data have come from a linear stochastic process. Often, a significant difference in the estimates of the correlation exponents, between the original and surrogate data sets, can be observed. In the case of the original data, a saturation of the correlation exponent is observed after a certain embedding dimension value (i.e., 7), whereas the correlation exponents computed for the surrogate data sets continue increasing with the increasing embedding dimension. The high significance values of the statistic indicate that the null hypothesis (the data arise from a linear stochastic process) can be rejected and hence the original data might have come from a nonlinear process.

The next important step is computing the Lyapunov's exponents, which are the dynamical invariants of a nonlinear system. In a general case, the orbits of chaotic attractors are unpredictable, but there is the limited predictability of chaotic physical system, which is defined by the global and local Lyapunov's exponents. In a chaos theory, the spectrum of the Lyapunov's exponents is considered a measure of the effect of perturbing the initial conditions of a dynamical system. Note that both positive and negative Lyapunov's exponents can coexist in a dissipative system, which is then chaotic. In fact, if one manages to derive the whole spectrum of the Lyapunov's exponents, other invariants of the system, i.e. Kolmogorov entropy and attractor's dimension can be found. The Kolmogorov entropy, K, measures the average rate at which information about the state is lost with time. An estimate of this measure is the sum of the positive Lyapunov's exponents. The inverse of the Kolmogorov entropy is equal to the average predictability. The estimate of the dimension of the attractor is provided by the Kaplan and Yorke conjecture:

$$d_L = j + \frac{\sum_{\alpha=1}^{j} \lambda_\alpha}{|\lambda_{j+1}|}, \tag{4}$$

where j is such that $\sum_{\alpha=1}^{j} \lambda_\alpha > 0$ and $\sum_{\alpha=1}^{j+1} \lambda_\alpha < 0$, and the Lyapunov's exponents λ_α are taken in descending order. There are a few approaches to computing the Lyapunov's exponents. One of them is based on the Jacobi matrix of system [30]. In the case where only observations are given and the system function is unknown, the matrix has to be estimated from the data. In this case, all the suggested methods approximate the matrix by fitting a local map to a sufficient number of nearby points. In our work we use the method with the linear fitted map proposed by Sano and Sawada [30], although the maps with higher order polynomials can be also used. To calculate the spectrum of the Lyapunov's exponents from the amplitude level data, one could determine the time delay τ and embed the data in the four-dimensional space. In this point it is very important to determine the Kaplan-York dimension and compare it with the correlation dimension, determined by the Grassberger-Procaccia algorithm. The estimations of the Kolmogorov entropy and average predictability can further show a limit, up to which the amplitude level data can be on average predicted. Surely, the important moment is a check of the statistical significance of results. It is worth to remind that results of state-space reconstruction are highly sensitive to the length of data set (i.e. it must be sufficiently large) as well as to the time lag and embedding dimension determined. Indeed, there are limitations on the applicability of chaos theory for observed (finite) time series arising from the basic assumptions that the time series must be infinite [31]. A finite and small data set may probably results in an underestimation of the actual dimension of the dynamical process. The statistical convergence tests together with surrogate data algorithm can provide the satisfactory significance of the investigated data regarding the state-space reconstruction. In Table 1 we present the main blocks of our universal chaos-geometric approach to computational studying a non-linear dynamics of the chaotic systems [6-19].

Table 1. A chaos-geometric approach to nonlinear analysis and prediction of chaotic dynamics of the complex systems

I. Preliminary study and assessment of the presence of chaos: 1. Test by Hottvod-Melben: $K \rightarrow 1$ – chaos; 2. Fourier decompositions, irregular nature of change – chaos; 3. Spectral analysis, Energy spectra statistics, the Wigner distribution, the spectrum of power, "Spectral rigidity";
⇓
II. The geometry of the phase space. Fractal Geometry: 4. Computation time delay τ using autocorrelation function or mutual information; 5. Determining embedding dimension d_E by the method of correlation dimension or algorithm of false nearest neighbouring points; 6. Calculation multi-fractal spectra. Wavelet analysis;
⇓
III. Prediction: 7. Computing global Lyapynov dimensions LE: λ_α; Kaplan-York dimension d_L, KE, average predictability measure Pr_{max}; 8. Determining the number of nearest neighbour points NN for the best prediction results; 9. Methods of nonlinear prediction. Neural network algorithm, the algorithm optimized trajectories, ...;

3. Results and conclusions

As illustration, below we present the results of computational studying the low- and high dimensional dynamics of a chaos generation in the semiconductor GaAs/GaAlAs laser with the retarded feedback. Fischer et al [32] have carried out the excellent experimental studying dynamics of a chaos generation in the semiconductor GaAs / GaAlAs Hitachi HLP1400 laser; an instability is generated by means of the retarded feedback during changing the control parameter such as the feedback strength μ (or in fact an injection current). Of course, depending on the system μ there is appeared a multi-stability of different states with the modulation period: $T_n=2\tau/(2n+1)$, n=0, 1,2,… The state of n = 0 is called as a ground one. With respect to the frequency modulation, other states are called as the third harmonic, fifth harmonic and so on. In the figure 1 we list the measured data on the time-dependent intensities for a semiconductor laser device with feedback: a) the time series, which illustrates a chaotic wandering between the ground state and the state of the third harmonic; b) the time series for a system in a state of the global chaotic attractor.

In Table 2 we present our original data on the correlation dimension d_2, the embedding dimension, computed on the basis of the false nearest neighboring points algorithm (d_N) with percentage of false neighbors (%) which are calculated for different lag times τ. The data are presented for two interesting regimes: I. chaos and II. hyperchaos. In Table 3 our computational data on the Lyapunov's exponents, Kaplan-York attractor dimensions, the Kolmogorov entropy K_{entr} are listed.

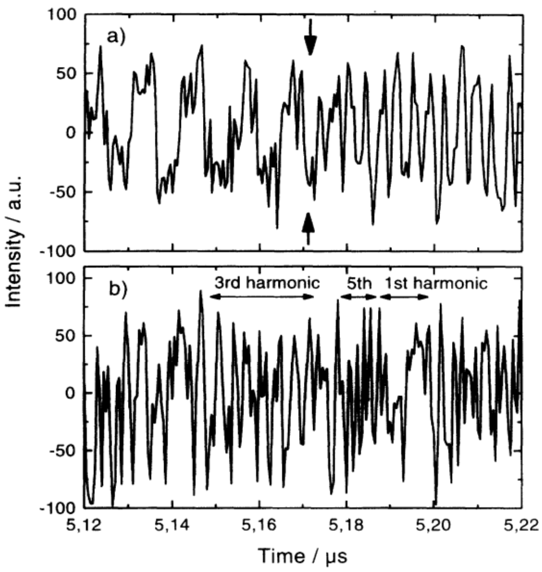

Figure 1. The time series of intensity in the GaAs/GaAlAs Hitachi HLP1400 laser (the measured data, from Ref. [32]).

Table 2. The correlation dimension d_2, the embedding dimension, computed on the basis of the false nearest neighboring points algorithm (d_N) with percentage of false neighbors (%) which are calculated for different lag times τ

Chaos regime (I)			Hyperchaos regime (II)		
τ	d_2	(d_N)	τ	d_2	(d_N)
58	3.4	5 (8.1)	67	8.4	11 (15)
6	2.2	4 (1.05)	10	7.4	8 (3.4)
8	2.2	4 (1.05)	12	7.4	8 (3.4)

Table 3. The Lyapunov's exponents: λ_1–λ_4, the Kaplan-York attractor dimension d_L and the Kolmogorov entropy K_{entr}

Regime	λ_1	λ_2	λ_3	d_L	K_{entr}
Chaos (I)	0.151	0.00001	-0.188	1.8	0.15
Hyperchaos (II)	0.517	0.192	-0.139	7.1	0.71

One can see that there are the Lyapunov's exponents positive and negative values. The resulting Kaplan-York dimensions in both cases are very similar to the correlation dimension, which is computed using the Grassberger-Procaccia algorithm [25]. The Kaplan-York dimension is less than the embedding dimension that confirms the correct choice of the latter. A scenario of chaos generation is in converting initially periodic states into individual chaotic states with increasing the parameter μ through a sequence of the period doubling bifurcations. Further there is appeared a global chaotic attractor after merging an individual chaotic attractors according to a few complicated scenario (see details in Refs. [13,32,33]).

To conclude, in this paper we have presented the results of computing nonlinear chaotic dynamics characteristics for the semiconductor GaAs/GaAlAs laser with retarded feedback system. The corresponding data have been obtained on the basis of using the advanced non-linear-analysis techniques such as a wavelet analysis, multi-fractal formalism, mutual information approach, correlation integral analysis and other methods. The correlation dimension method provided a low (or high-) fractal-dimensional attractor thus suggesting a possibility of an existence of the chaotic behaviour. The method of surrogate data, for detecting nonlinearity, provided significant differences in the correlation exponents between the original data series and the surrogate data sets. This finding indicates that the null hypothesis (linear stochastic process) can be rejected. It has been finally confirmed that the studied laser system dynamics exhibit a nonlinear behaviour with elements of the low-and high-dimensional chaos.

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