Anisotropic solutions in modified gravity

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Abstract. Anisotropic but homogeneous, shear-free cosmological models with imperfect matter sources in $f(R)$ gravity are investigated. The relationship between the anisotropic stresses and the electric part of the Weyl tensor, as well as their evolutions in orthogonal $f(R)$ models, is explored. The late-time behaviour of the de Sitter universe (as an example of a locally rotationally symmetric spacetimes in orthonormal frames) in $f(R)$ gravity is examined. By taking initial conditions for the expansion, acceleration and jerk parameters from observational data, numerical integrations for the evolutionary behavior of the Universe in the Starobinsky model of $f(R)$ have been carried out.

1. Introduction

Despite General Relativity's (GR) great successes in explaining many cosmological and astrophysical scenarios, it miserably fails to provide:

- an elegant solution to the early and current accelerated expansion phases of the Universe,
- the mechanism for dark matter production (if one is convinced that dark matter exists),
- a consistent framework to combine gravity with the other three forces of nature.

As a result, recently a large number of alternative, modified or generalized propositions to GR have emerged, one of which involves the inclusion of higher-order curvature invariants in the Einstein-Hilbert action [1, 2, 3, 4, 5]:

$$A = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m]\ , \tag{1}$$

where $f(R)$ is some differentiable function in the Ricci curvature scalar R, g is the determinant of the spacetime metric g_{ab}, $\mathcal{L}_m$ is the standard matter Lagrangian, and the natural units ($\hbar = c = k_B = 8\pi G = 1$) have been used. The generalised Einstein Field Equations (EFEs), obtained from variational principles, is given by

$$G_{ab} = \tilde{T}^m_{ab} + T^R_{ab} \equiv T_{ab}\ , \tag{2}$$

where the modified matter energy-momentum tensor is given by

$$\tilde{T}^m_{ab} \equiv \frac{T^m_{ab}}{f'}\ , \qquad T^m_{ab} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{ab}}\ , \tag{3}$$

and the energy-momentum tensor of the *curvature fluid* can be defined as

$$T^R_{ab} \equiv \frac{1}{f'}\left[\tfrac{1}{2}(f-Rf')g_{ab}+\nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f'\right] \ ;\text{with } f'\equiv df/dR, \text{etc} \ . \tag{4}$$

The energy-momentum tensor of standard matter is given by

$$T^m_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} = \mu_m u_a u_b + p_m h_{ab} + q^m_a u_b + q^m_b u_a + \pi^m_{ab} \ , \tag{5}$$

where we have defined

- $\mu_m \equiv T^m_{ab}u^a u^b$ as the energy density
- $p_m \equiv \frac{1}{3}(T^m_{ab}h^{ab})$ as the isotropic pressure
- $q^m_a \equiv -T^m_{bc}u^b h^c_a$ as the heat flux
- $\pi^m_{ab} \equiv T^m_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}$ as the anisotropic pressure

of matter. The four-vector $u^a \equiv \frac{dx^a}{dt}$ is the normalized 4-velocity of fundamental observers comoving with the fluid. The covariant time derivative OF any tensor $S^{a..b}_{c..d}$ along an observer's worldlines is defined as

$$\dot{S}^{a..b}_{c..d} = u^e\nabla_e S^{a..b}_{c..d} \ , \tag{6}$$

whereas the fully orthogonally projected covariant derivative for any tensor $S^{a..b}_{c..d}$ is given by

$$\tilde{\nabla}_e S^{a..b}_{c..d} = h^a_f h^p_c ... h^b_g h^q_d h^r_e \nabla_r S^{f..g}_{p..q} \ , \tag{7}$$

with total projection on all the free indices. Here r$h_{ab} \equiv g_{ab} + u_a u_b$ is known as the projection tensor. We extract the orthogonally projected symmetric trace-free part of vectors and rank-2 tensors using

$$V^{\langle a\rangle} = h^a_b V^b \ , \ S^{\langle ab\rangle} = \left[h^{(a}_c h^{b)}_d - \tfrac{1}{3}h^{ab}h_{cd}\right]S^{cd} \ , \tag{8}$$

and the volume element for the restspaces orthogonal to u^a is given by [6]

$$\varepsilon_{abc} = u^d\eta_{dabc} = -\sqrt{|g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]}, \ \varepsilon_{abc}u^c = 0, \tag{9}$$

where η_{abcd} is the 4-dimensional volume element satisfying the conditions

$$\eta_{abcd} = \eta_{[abcd]} = 2\varepsilon_{ab[c}u_{d]} - 2u_{[a}\varepsilon_{b]cd}. \tag{10}$$

The covariant spatial divergence and curl of vectors and rank-2 tensors are given as [7]

$$\text{div}V = \tilde{\nabla}^a V_a \ , \quad (\text{div}S)_a = \tilde{\nabla}^b S_{ab} \ , \quad curl V_a = \varepsilon_{abc}\tilde{\nabla}^b V^c \ , \quad curl S_{ab} = \varepsilon_{cd(a}\tilde{\nabla}^c S_{b)}{}^d \ . \tag{11}$$

In this formalism, u^a can be split into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \varepsilon_{abc}\omega^c, \tag{12}$$

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a}u_{b\rangle}$ and $\omega^a \equiv \varepsilon^{abc}\tilde{\nabla}_b u_c$ are the 4-acceleration, (volume) expansion, shear and vorticity of the fluid. The thermodynamic quantities for the curvature fluid can be defined similarly to the standard matter ones:

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf'-f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right] , \tag{13}$$

$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f-Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right)\right] , \tag{14}$$

$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a \dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right] , \tag{15}$$

$$\pi_{ab}^R = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}f''\right] , \tag{16}$$

whereas the total thermodynamics of the matter-curvature fluid composition is described by

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R , \quad p \equiv \frac{p_m}{f'} + p_R , \quad q_a \equiv \frac{q_a^m}{f'} + q_a^R , \quad \pi_{ab} \equiv \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R . \tag{17}$$

The Weyl conformal curvature tensor

$$C^{ab}{}_{cd} \equiv R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]} \tag{18}$$

can be split into its "gravito-electric" (GE) and "gravito-magnetic" (GM) parts, respectively:

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} = \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d. \tag{19}$$

The GE and GM components influence the motion of matter and radiation through the geodesic deviation for timelike and null-vector fields, respectively [6]. The GM has no Newtonian analogue, and is responsible for gravitational radiation.

By covariantly $1+3$-splitting the Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^e = 0 , \quad (\nabla_a\nabla_b - \nabla_b\nabla_a)u_c = R_{abc}{}^d u_d \tag{20}$$

for the total fluid 4-velocity u^a, we obtain the following field (propagation and constraint) equations. The propagation equations uniquely determine the covariant variables on some initial hypersurface S_0 at $t=t_0$:

$$\dot{\mu}_m = -(\mu_m+p_m)\Theta - \tilde{\nabla}^a q_a^m - 2A_a q_m^a - \sigma_b^a\pi_{a,m}^b , \tag{21}$$

$$\dot{\mu}_R = -(\mu_R+p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q_a^R - 2A_a q_R^a - \sigma_b^a\pi_{a,R}^b , \tag{22}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu+3p) + \tilde{\nabla}_a A^a - A_a A^a - \sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a , \tag{23}$$

$$\dot{q}_a^m = -\tfrac{4}{3}\Theta q_a^m - (\mu_m+p_m)A_a - \tilde{\nabla}_a p_m - \tilde{\nabla}^b\pi_{ab}^m - \sigma_a^b q_b^m - A^b\pi_{ab}^m - \varepsilon_{abc}\omega^b q_m^c , \tag{24}$$

$$\dot{q}_a^R = -\tfrac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b\pi_{ab}^R - \sigma_a^b q_b^R - (\mu_R+p_R)A_a - A^b\pi_{ab}^R - \varepsilon_{abc}\omega^b q_R^c , \tag{25}$$

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\varepsilon_{abc}\tilde{\nabla}^b A^c + \sigma_a^b\omega_b , \tag{26}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - \sigma^c_{\langle a}\sigma_{b\rangle c} - \omega_{\langle a}\omega_{b\rangle} , \tag{27}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}\left(\mu+p\right)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle}$$

$$+ 3\sigma_a^{\langle c}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) - A_{\langle a}q_{b\rangle} + \varepsilon_{cd\langle a}\left[2A^c H^d_{b\rangle} + \omega^c(E^d_{b\rangle} + \tfrac{1}{2}\pi^d_{b\rangle})\right] , \tag{28}$$

$$\dot{H}_{ab} = -\Theta H_{ab} - \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle}$$
$$+ 3\sigma_a^{\langle c}H_{b\rangle c} + \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} - \varepsilon_{cd\langle a}\left[2A^c E^d_{b\rangle} - \tfrac{1}{2}\sigma^c_{b\rangle}q^d - \omega^c H^d_{b\rangle}\right] . \tag{29}$$

Restrictions on the initial data to be specified are provided by the constraint equations:

$$(C^1)_a := \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \varepsilon_{abc}\left(\tilde{\nabla}^b\omega^c + 2A^b\omega^c\right) + q_a = 0 , \tag{30}$$

$$(C^2)_{ab} := \varepsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} - 2A_{\langle a}\omega_{b\rangle} = 0 , \tag{31}$$

$$(C^3)_a := \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \varepsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right]$$
$$+ 3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right) = 0 , \tag{32}$$

$$(C^4)_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a$$
$$- \tfrac{1}{2}\sigma^b_a q_b - 3\omega^b H_{ab} - \varepsilon_{abc}[\sigma^{bd}H^c_d - \tfrac{3}{2}\omega^b q^c] = 0 , \tag{33}$$

$$(C^5) := \tilde{\nabla}^a\omega_a - A_a\omega^a = 0 , \tag{34}$$

and the Gauß-Codazzi equations, given by

$$\tilde{R}_{ab} + \dot{\sigma}_{\langle ab\rangle} + \Theta\sigma_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} - A_{\langle a}A_{b\rangle} - \pi_{ab} - \frac{1}{3}\left(2\mu - \frac{2}{3}\Theta^2\right)h_{ab} = 0 . \tag{35}$$

Here $\tilde{R}_{ab}$ is the Ricci tensor on 3-D spatial hypersurfaces, the trace of which gives the corresponding (3-curvature) Ricci scalar: $\tilde{R} = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2$. The constraint equations must remain satisfied on any hypersurface S_t for all comoving time t.

In orthogonal cosmological models, the matter energy density μ_m and isotropic pressure p_m are measured by an observer moving with the velocity u^a. These models are characterised by the matter energy-momentum tensor representing an anisotropic fluid without heat fluxes [8]

$$T^m_{ab} = \mu_m u_a u_b + p_m h_{ab} + \pi^m_{ab} , \tag{36}$$

and by an irrotational and non-accelerated flow of the vector field u^a, $\omega_a = 0 = A_a$. The revised evolution and constraint equations for orthogonal models are now given by

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \sigma^a_b\pi^b_{a,m} , \tag{37}$$

$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q^R_a - \sigma^a_b\pi^b_{a,R} , \tag{38}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) - \sigma_{ab}\sigma^{ab} , \tag{39}$$

$$\dot{q}^R_a = -\tfrac{4}{3}\Theta q^R_a + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b\pi^R_{ab} - \sigma^b_a q^R_b , \tag{40}$$

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} - \sigma^c_{\langle a}\sigma_{b\rangle c} , \tag{41}$$

$$\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} = \varepsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle}$$
$$+ 3\sigma_a^{\langle c}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) , \tag{42}$$

$$\dot{H}_{ab} = -\Theta H_{ab} - \varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c\pi^d_{b\rangle} + 3\sigma_a^{\langle c}H_{b\rangle c}$$
$$+ \tfrac{1}{2}\varepsilon_{cd\langle a}\sigma^c_{b\rangle}q^d_R , \tag{43}$$

$$(C^{*1})_a := \tilde{\nabla}^b \sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a \Theta + q^R_a = 0\ , \tag{44}$$

$$(C^{*2})_{ab} := \varepsilon_{cd(a}\tilde{\nabla}^c \sigma_{b)}{}^d - H_{ab} = 0\ , \tag{45}$$

$$(C^{*3})_a := \tilde{\nabla}^b H_{ab} + \varepsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c_R + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] = 0\ , \tag{46}$$

$$(C^{*4})_a := \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b \pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a \mu + \tfrac{1}{3}\Theta q^R_a - \tfrac{1}{2}\sigma^b_a q_b - \varepsilon_{abc}\sigma^{bd} H^c_d = 0\ . \tag{47}$$

We notice that a new constraint

$$(C^{*5})_a := \tilde{\nabla}_a p_m + \tilde{\nabla}^b \pi^m_{ab} = 0 \tag{48}$$

comes out of equation (24) as a result of the orthogonality assumption.

2. Shear-free anisotropic models with an imperfect fluid

For imperfect fluids, the the thermodynamic evolution equation for the anisotropic pressure is given by [9, 10]

$$\tau \dot{\pi}_{ab} + \pi_{ab} = -\lambda \sigma_{ab}\ . \tag{49}$$

Here τ and λ are, respectively, relaxation and viscosity parameters. For negligible τ and a positive constant λ, the equation of state between the shear and anisotropic pressure is given by [8]

$$\pi_{ab} = -\lambda \sigma_{ab}\ . \tag{50}$$

Making use of equations (17) and (16), equation (50) can now be rewritten:

$$\pi^m_{ab} + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R = \sigma_{ab}\left(\dot{R}f'' - \lambda f'\right)\ . \tag{51}$$

This implies that shear-free in the case of shear-free fluid spacetimes, the above equation and the Gauß-Codazzi equations (35) simplify, respectively, to

$$\pi^m_{ab} = -f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R - f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R\ , \tag{52}$$

$$\tilde{R}_{ab} - \tfrac{1}{3}\tilde{R}h_{ab} = \pi_{ab} = \frac{1}{f'}\left(\pi^m_{ab} + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R\right)\ . \tag{53}$$

These results show that even in the case of vanishing anisotropic pressure from matter, spacetime geometries are not necessarily of constant curvature and hence not necessarily FLRW universes. If we allow the matter anisotropic pressure to be nonzero despite a vanishing shear, constant-curvature models are allowed, unlike in GR, provided

$$f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R = 0\ . \tag{54}$$

In the case of shear-free fluid spacetimes, we notice from equation (41) that the tidal effect (represented by the EM component of the Wely tensor) on the anisotropic stress is given by

$$\pi_{ab} = 2E_{ab}\ . \tag{55}$$

Thus the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other, a necessary and sufficient condition for the shear to remain zero if initially vanishing [8, 11].

For nonzero, second-order shear contributions, equation (41) can be approximated by

$$\dot{\sigma}_{ab} \approx -\tfrac{2}{3}\Theta\sigma_{ab}\,, \quad \left(\sigma^2\right)^{\cdot} \approx -\tfrac{4}{3}\Theta\sigma^2. \tag{56}$$

This clearly shows that small perturbations of shear are damped in the class of orthogonal $f(R)$ models in $f(R)$. In agreement with GR results [8], these models are stable if expanding.

Shear-free orthogonal models satisfying equation (55) are purely EM, i.e., $H_{ab} = 0$. Thus, equation (43) reduces to an identity

$$\varepsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} = \tfrac{1}{2}\varepsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle}\,, \tag{57}$$

whereas using equations (42) and (47), it can be shown that the evolution and divergence of the EM Weyl tensor are given by

$$\dot{E}_{ab} = -\tfrac{2}{3}\Theta E_{ab} - \tfrac{1}{4}\tilde{\nabla}_{\langle a}q^R_{b\rangle}\,, \quad \tilde{\nabla}^b E_{ab} = \tfrac{1}{6}\left(\tilde{\nabla}_a\mu - \tfrac{1}{3}\Theta q^R_a\right)\,. \tag{58}$$

The decaying of the EM Weyl tensor, and hence of the anisotropic stress tensor, with the expansion is demonstrated by the relation

$$\left(E^2\right)^{\cdot} = -\tfrac{4}{3}\Theta E^2 - \frac{1}{8}\left(\tilde{\nabla}_{\langle a}q^R_{b\rangle}E^{ab} + \tilde{\nabla}^{\langle a}q^{b\rangle}_R E_{ab}\right)\,, \quad E^2 \equiv E_{ab}E^{ab}. \tag{59}$$

3. Illustration using the Starobinsky $f(R)$ model

As a simple illustration, we will try to integrate the Friedmann equation in locally rotationally symmetric spacetimes

$$3\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu, \tag{60}$$

provided the barotropic EoS , $p_m = (\gamma_m - 1)\mu_m$. If we rewrite (60) using equations (17) and (37) (for shear-free cases, of course), we obtain the model-dependent equation

$$3\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu^0_m a^{-3\gamma_m} + \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \frac{3\dot{a}}{a}f''\dot{R}\right] \tag{61}$$

where μ^0_m is the matter density at the time $t = t_0$ and γ_m is the EoS parameter for the matter content. A qualitative analysis of the late-time behavior of the solutions for (61) in flat $k = 0$ spacetimes without matter gives a de Sitter (dS) solution, with $R = 6H_0^2$ and equation (61) solves to

$$H_0^2 = \frac{1}{6f'}(Rf' - f). \tag{62}$$

The Friedmann equation (60) for generic $f(R)$ model is, in general, a fourth-order ordinary differential equation (ODE). There are no known exact solutions for the full evolution history, but the equation can be solved numerically (such as in terms of quadratures) given appropriate initial conditions. For the purpose of our illustration, if we choose the Starobinsky model,

$$f(R) = R + \alpha R^2\,, \tag{63}$$

equation (61) reduces to the following differential equation:

$$3\frac{\dot{a}^2}{a^2} = \mu^0_m a^{-3\gamma_m} + \frac{\alpha}{2}\frac{R^2 - 12H\dot{R}}{1 + 2\alpha R}\,, \qquad R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\,. \tag{64}$$

This is a third-oder ODE in $a(t)$. To solve it, let us use the cosmological initial conditions (ICs) for the Hubble H, deceleration q, jerk j, and snap s parameters:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}, \qquad j \equiv \frac{a^2}{\dot{a}^3}\frac{d^3a}{dt^3}, \qquad s \equiv \frac{a^3}{\dot{a}^4}\frac{d^4a}{dt^4} \tag{65}$$

evaluated at the present time $t = t_0$, such that

$$a(0) = a_0 = 1 \ , \qquad \dot{a}(0) = H_0 a_0 \ , \qquad \ddot{a}(0) = -H_0^2 a_0 q_0 \ , \qquad \frac{d^3a}{dt^3}(0) = H_0^3 j_0 a_0^{-1} \ . \tag{66}$$

A series solution using these cosmographic parameters in equation (64), evaluated at $t = t_0$ can be given by

$$\begin{aligned}
a(t) &= 1 + H_0\,(t - t_0) - 1/2\,{H_0}^2 q_0\,(t - t_0)^2 \\
&- \frac{1}{216}\,\frac{\left(-3\,{H_0}^2 + 54\,{H_0}^4\alpha + \mu_m + 12\,\alpha\,\mu_m {H_0}^2 - 12\,\alpha\,\mu_m^0 {H_0}^2 q_0 + 18\,\alpha\,{H_0}^4 {q_0}^2 + 36\,{H_0}^4\alpha\,q_0\right)}{\alpha\,H_0}\,(t - t_0)^3 \\
&+ \frac{1}{2592}\,\frac{(t - t_0)^4}{\alpha\,{H_0}^2} \times \Big(9\,{H_0}^4 + 162\,{H_0}^6\alpha - 12\,\mu_m^0 {H_0}^2 + 18\,{H_0}^4\alpha\,\mu_m^0 + 108\,{H_0}^4\alpha\,\mu_m^0 q_0 - 54\,{H_0}^6\alpha\,{q_0}^2 \\
&+ 324\,{H_0}^6\alpha\,q_0 - 108\,\alpha\,{H_0}^6 {q_0}^3 - 6\,\mu_m^0 {H_0}^2 q_0 + 9\,\mu_m^0\gamma_m {H_0}^2 + {\mu_m}^2 + 90\,\alpha\,\mu_m^0 {H_0}^4 {q_0}^2 - 12\,\alpha\,{\mu_m^0}^2 {H_0}^2 q_0 \\
&+ 12\,\alpha\,{\mu_m}^2 {H_0}^2 + 108\,\mu_m^0\gamma_m {H_0}^4\alpha - 108\,\mu_m^0\gamma_m {H_0}^4\alpha\,q_0\Big) + O\left[(t - t_0)^5\right]
\end{aligned} \tag{67}$$

and can be used to check observational constraints. If we solve equation (64) numerically and plot the solutions versus time, we notice from figure 1 that H is an oscillatory function which can be identified in the late-time as the ΛCDM era.

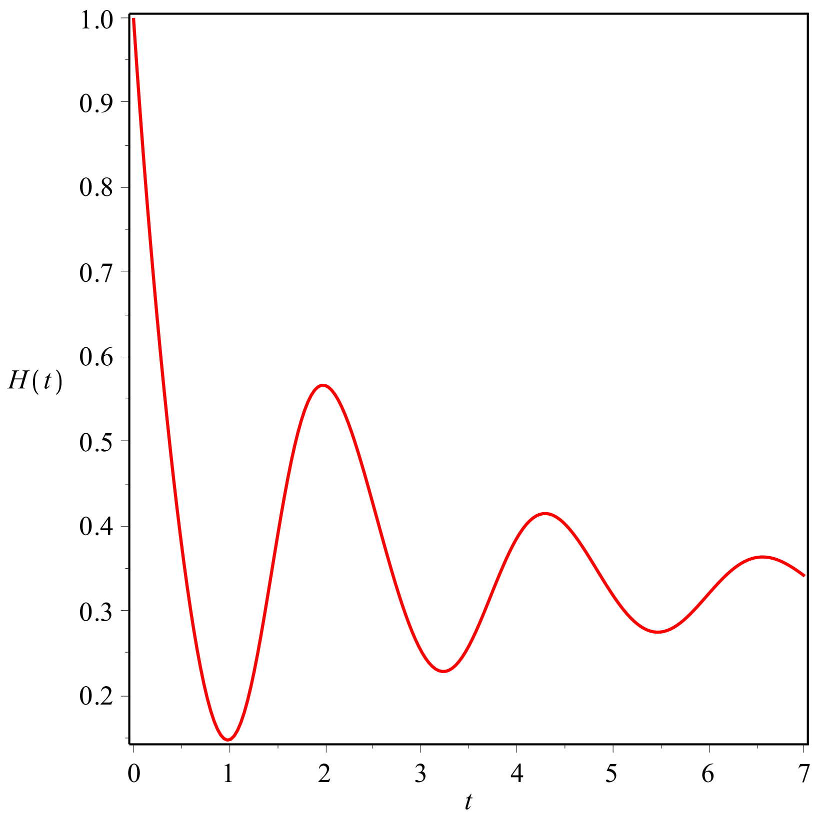

Figure 1. Numerical solution for $H(t)$. Model: $\alpha = 0.02\ , a_0 = H_0 = 1\ , q_0 = -0.7$. The solution, which is oscillatory in nature, can be identified in the late-time as the ΛCDM era.

The Hubble parameter and its first, second and third derivatives of H are plotted in figure 2.

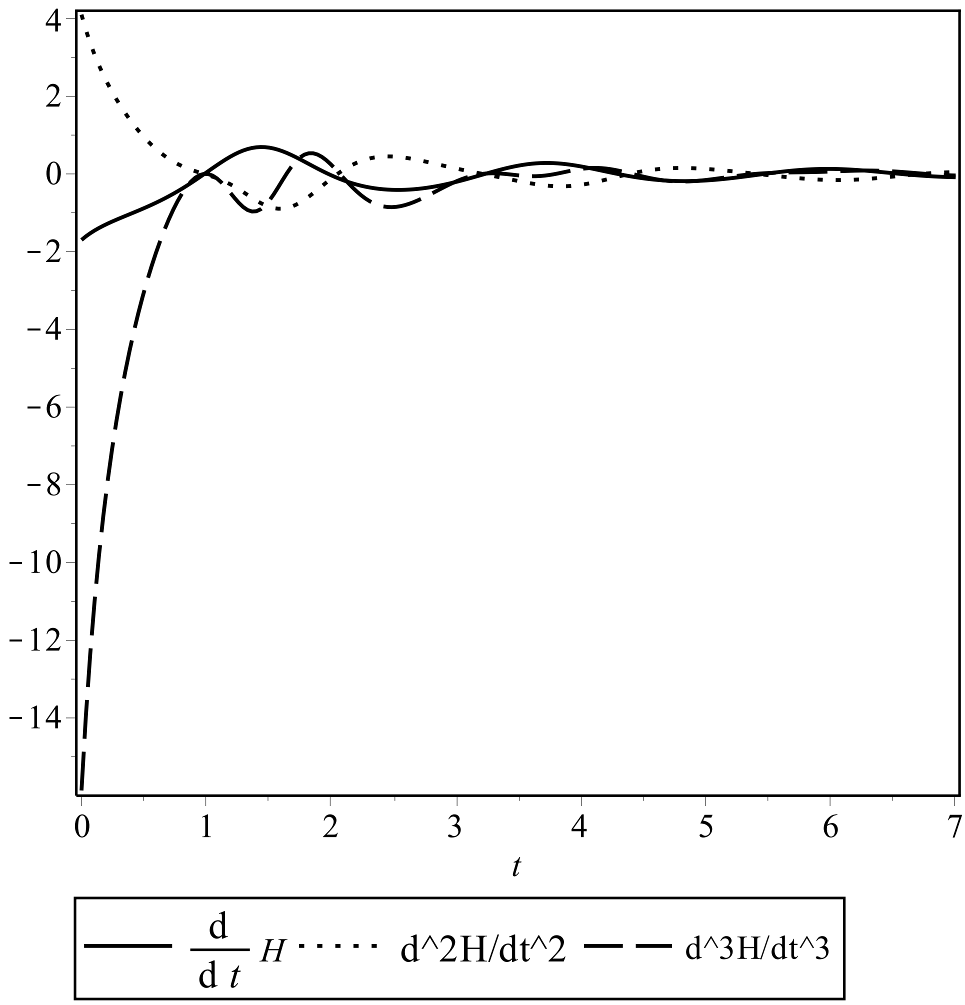

Figure 2. Numerical solution for the first three derivatives of the Hubble parameter. Note the singularity-free nature of the solution, as none of the higher derivatives of H diverges.

No higher derivatives of H diverges and, therefore, our solution is singularity free. Specializing to dust models, i.e., $\Omega_m^0 \equiv \frac{\mu_m^0}{3H_0^2} = 0.3$, $\gamma_m = 1$, we plot the phase portrait for the Starobinsky model in figure 3.

The model is well established as an attractor. Figure 4 shows that the model is a late-time or asymptotic attractor, the solutions to the equations of motion have a generic form independent of the initial conditions.

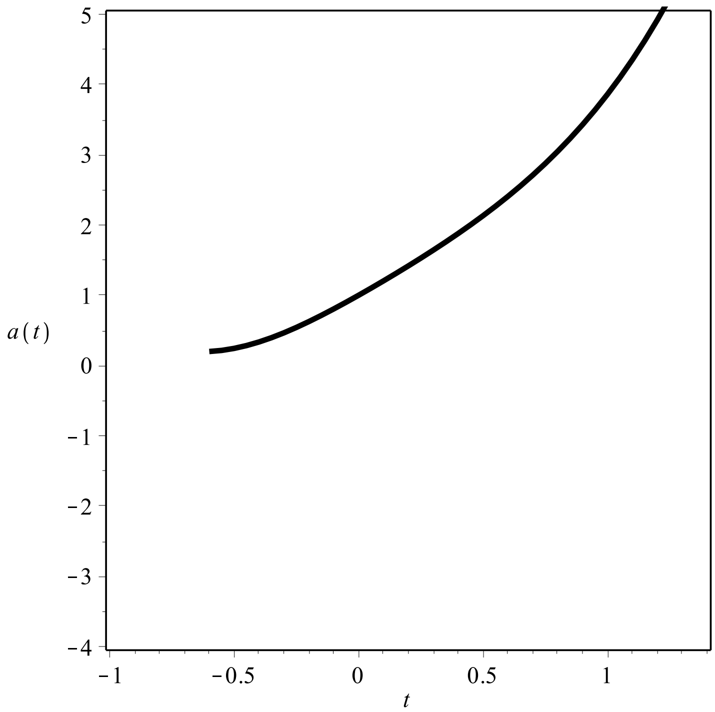

Figure 3. The phase portrait for Starobinsky's dust model. The scale factor $a(t)$ is a monotonically increasing function of cosmic time t.

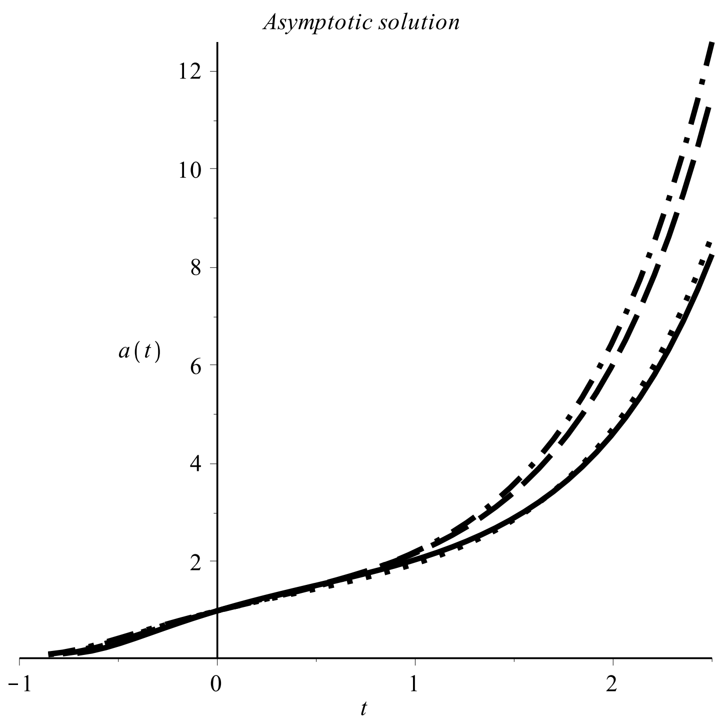

Figure 4. Late-time asymptotic attractors for Starobinsky's model of gravitation. The solutions to the equations of motion have a generic form independent of the initial conditions.

4. Conclusion

In this work we have looked at classes of shear-free anisotropic cosmological spacetimes in $f(R)$ gravity. Specializing to orthogonal models with irrotational and non-accelerated fluid flows without heat fluxes, we have derived the relationship between the anisotropic stresses and electric part of the Weyl tensor, which is the necessary and sufficient condition for the shear to be vanishing forever if vanishing initially.

Moreover, we have shown that within the class of orthogonal $f(R)$ models, small perturbations of shear are damped. Considering a subclass of locally rotationally symmetric spacetimes with barotropic equations of state, we have shown that the late-time behaviour of the dS universe in $f(R)$ gravity should satisfy equation (62).

Finally we have provided a power-series solution for $a(t)$ and studied the behavior of the expansion parameter $H(t)$ by numerically integrating the Friedmann equation (64), where the initial conditions for H_0 , q_0 and j_0 are taken from observational data.

A full computational implementation of the field equations under realistic initial conditions is left for a subsequent work.

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