Non-locality without inequality and generalized non-local theory

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Abstract. We find non-local but non-signaling probabilities satisfying the 'nonlocality without inequality' arguments for multiple two-level systems. Maximum probability of success of these arguments are obtained in the framework of a generalized nonlocal theory. Interestingly, for two two-level systems, the probability of success of these arguments converge to a common maximum in this framework. This is in sharp contrast with the quantum case, where for such systems, Cabello's argument succeeds more than that of Hardy's. We also find that the maximum probability of success of Hardy's argument is the same for both the two and three two-level systems in the framework of this more generalized theory.

1. Introduction

There exist correlations between quantum systems which no local-realistic theory can reproduce. This was first shown by Bell by means of an inequality, known as Bell's inequality [1]. Later, Hardy [2] gave an argument which also reveals the non-local character of Quantum Mechanics, but this argument, unlike Bell's argument, does not use inequalities involving expectation values. Afterwards, Cabello [3] introduced another logical structure to prove Bell's theorem without inequality. Though, Cabello's logical structure was originally proposed for establishing nonlocality for three particle states, but it was later exploited to establish nonlocality for a class of two-qubit mixed entangled state [4]. It is noteworthy here that in contrast, there is no two-qubit mixed state which shows Hardy type nonlocality [5] whereas almost all pure entangled states of two-qubits do so (maximally entangled states are the exception)[6, 7]. Likewise, for almost all two-qubit pure entangled states other than maximally entangled one, Cabello's nonlocality argument runs, but, intesestingly, for these states, the maximum probabilty of success of this argument is more than that of the Hardy's [8].

Although no local-realistic theory can reproduce quantum correlations still these corre- lations cannot be exploited to communicate with a speed greater than that of the light in vacuum. But quantum theory is not the only non-local theory consistent with the relativistic causality [9]. Theories which predict non-local correlations and hence permit violation of Bell's inequality but are constrained with the no signalling condition are called Generalized non-local theory (GNLT). In recent years there has been an increasing interest in GNLT [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In general, quantum theory has been studied in the background of classical theory which is comparatively restrictive. The new idea is to study quantum theory from outside i.e., starting from a general family of theories, and to study properties common to all [13]. This might help in a better understanding of quantum non-locality.

In this paper, we study the non-locality arguments in the framework of GNLT. We find that probability of success of both the Hardy's and the Cabello's nonlocality arguments converge to a common maximum 0.5 for two two-level systems unlike the quantum case where they are respectively 0.09 [6, 20] and 0.11 (approx) [8]. We, further find that 0.5 is also the maximum probability of success of Hardy's argument for three two-level systems in GNLT, the corresponding qunatum maximum is known to be 0.125 [21].

2. Nonlocality arguments for two two-level systems and the generalized non local theory

In the framework of a generalized probabilistic theory, consider a physical system consisting of two subsystems shared between two far seperated observers, Alice and Bob. Assume that Alice can run the experiments of measuring any one (chosen at random) of the two $\{-1,+1\}$-valued random variables A and $A^{'}$ corresponding to her subsystem whereas Bob can run the experiments of measuring any one (chosen at random) of the two $\{-1,+1\}$-valued random variables B and $B^{'}$ corresponding to his subsystem. Consider now the following four conditions:

$$\left.\begin{array}{lcl} \text{Prob}(A=+1,\ B=+1) & = & 0, \\ \text{Prob}(A^{'}=+1,\ B=-1) & = & 0, \\ \text{Prob}(A=-1,\ B^{'}=+1) & = & 0, \\ \text{Prob}(A^{'}=+1,\ B^{'}=+1) & = & q \end{array}\right\} \text{(with } q>0\text{)}. \tag{1}$$

The above four conditions together form the basis of Hardy's nonlocality argument. These four cannot be fulfilled simultaneously in the framework of a local-realistic theory. To see this let us start with the fourth condition in equation (1) which implies that (i) there is a non-zero probability (which is q here) of simultaneous occurrence of $A^{'}=+1$ and $B^{'}=+1$. This will then imply that (according to classical probability theory) (ii) if Alice chooses to perform the measurement of $A^{'}$, there will be a non-zero probability (which should be at least q) of getting the value $+1$, and similarly, (iii) if Bob chooses to perform the measurement of $B^{'}$, there will be a non-zero probability (which should be at least q) of getting the value $+1$. Now the 3rd condition in equation (1) implies that if Bob chooses for the measurement of $B^{'}$ and Alice chooses for A, she is bound to get the value $+1$ whenever Bob gets the value $+1$ (which Bob can indeed get with a non-zero probability, according to (iii)). Similarly the 2nd condition in equation (1) implies that if Alice chooses to the measurement of $A^{'}$ and Bob chooses to perform B, he is bound to get the value $+1$ whenever Alice gets the value $+1$ (which Alice can indeed get with a non-zero probability, according to (ii)). So $A=+1$ is a 'reality' of A while $B=+1$ is also a 'reality' of B, according to EPR[23]. Now, invoking 'locality', $A=+1$ and $B=+1$ is a joint 'reality' of the composite system. And, according to condition (i) (again, using classical probability arguments), the joint probability of occurrence of $A=+1$, $B=+1$ must be at least q. This contradicts the first condition of equation (1). On the contrary almost all pure entangled state of two-qubit (except the maximally entangled one) satisfy the above four conditions simultaneously for suitable choices of observables[7] with a maximum probability of success (q_{max}) equal to 9% (approx) [6, 20].

Cabello's conditions result by replacing the right hand side of the first condition of (1) with a nonzero probability p with $p<q$, and keeping the remaining three conditions intact. As seen above, for local-realistic states to satisfy all the four conditions simultaneously, the joint probability of occurrence of $A=+1$, $B=+1$ must be at least equal to q and thus these conditions are also incompatible with local-realism. But, almost all pure entangled states of two-qubits (except their maximally entangled states) satisfy these with a maximum success probability equal to 11% (approx)[8].

We now study the character of a set of sixteen non-signalling joint probabilities $P(i_m,j_n)$ where $P(i_m,j_n)$ denotes the probability of getting the output as $i_m\in\{+1,-1\}$ and $j_n\in$

$\{+1,-1\}$ in a measurement for $M \in \{A, A^{'}\}$ by Alice and $N \in \{B, B^{'}\}$ by Bob. These sixteen joint probabilities are given below:

$$\begin{aligned}
&\mathrm{Prob}(A=+1,B=+1)=p_1,\ \mathrm{Prob}(A=+1,B=-1)=p_2,\\
&\mathrm{Prob}(A=-1,B=+1)=p_3,\ \mathrm{Prob}(A=-1,B=-1)=p_4,\\
\\
&\mathrm{Prob}(A^{'}=+1,B=+1)=p_5,\ \mathrm{Prob}(A^{'}=+1,B=-1)=p_6,\\
&\mathrm{Prob}(A^{'}=-1,B=+1)=p_7,\ \mathrm{Prob}(A^{'}=-1,B=-1)=p_8,\\
\\
&\mathrm{Prob}(A=+1,B^{'}=+1)=p_9,\ \mathrm{Prob}(A=+1,B^{'}=-1)=p_{10},\\
&\mathrm{Prob}(A=-1,B^{'}=+1)=p_{11},\ \mathrm{Prob}(A=-1,B^{'}=-1)=p_{12},\\
\\
&\mathrm{Prob}(A^{'}=+1,B^{'}=+1)=p_{13},\ \mathrm{Prob}(A^{'}=+1,B^{'}=-1)=p_{14},\\
&\mathrm{Prob}(A^{'}=-1,B^{'}=+1)=p_{15},\ \mathrm{Prob}(A^{'}=-1,B^{'}=-1)=p_{16}.
\end{aligned} \tag{2}$$

Other than being members of the interval $[0,1]$, these probabilities must satisfy the normalization condition:

$$p_1+p_2+p_3+p_4=1, \tag{3}$$

$$p_5+p_6+p_7+p_8=1, \tag{4}$$

$$p_9+p_{10}+p_{11}+p_{12}=1, \tag{5}$$

$$p_{13}+p_{14}+p_{15}+p_{16}=1. \tag{6}$$

The no-signaling constraint implies that if Alice measures for A (or $A^{'}$), the individual probabilities for the outcomes $A=+1$ (or $A^{'}=+1$) and $A=-1$ (or $A^{'}=-1$) must be independent of whether Bob chooses to measure for B or $B^{'}$ and similar should be the case for Bob too. So, the condition for causality to hold is given by:

$$p_1+p_2=p_9+p_{10}, \tag{7}$$

$$p_3+p_4=p_{11}+p_{12}, \tag{8}$$

$$p_5+p_6=p_{13}+p_{14}, \tag{9}$$

$$p_7+p_8=p_{15}+p_{16}, \tag{10}$$

$$p_1+p_3=p_5+p_7, \tag{11}$$

$$p_2+p_4=p_6+p_8, \tag{12}$$

$$p_9+p_{11}=p_{13}+p_{15}, \tag{13}$$

$$p_{10}+p_{12}=p_{14}+p_{16}. \tag{14}$$

We further assume that these probabilities respect the Hardy-type non-locality conditions:

$$p_1=0,\ p_6=0,\ p_{11}=0,\ p_{13}=q. \tag{15}$$

Using equation (15) into equation (7), we get

$$p_2 \geq p_9. \tag{16}$$

Using equation (15) into equation (11), we get

$$p_3 \geq p_5. \tag{17}$$

Using equation (15) into equation (3), we get

$$1 = p_2 + p_3 + p_4 \geq p_2 + p_3 \geq p_5 + p_9, \tag{18}$$

using equations (16) and (17). Using equation (15) into equations (9) and (13), we get

$$p_5 + p_9 = 2q + p_{14} + p_{15} \geq 2q. \tag{19}$$

Using equations (18) and (19), we get

$$1 \geq p_5 + p_9 \geq 2q.$$

Thus we have

$$q \leq \frac{1}{2}. \tag{20}$$

If we now follow the argument, beginning at equation (16) and ending at equation (20), it can be easily shown that for $q = 1/2$, there is a *unique* solution for the above-mentioned sixteen joint probabilities satisfying simultaneously all the conditions (3) to (15):

$$p_2 = p_3 = p_5 = p_8 = p_9 = p_{12} = p_{16} = \frac{1}{2} \text{ and } p_4 = p_7 = p_{10} = p_{14} = p_{15} = 0. \tag{21}$$

Thus we see that above-mentioned sixteen probabilities will be *non-local* as well as *non-signaling* iff $0 < q \leq 1/2$.

If, inplace of Hardy's conditions, we put the Cabello's conditions $p_1 = p$, $p_6 = 0$, $p_{11} = 0$ and $p_{13} = q$ in equations (3)-(14), we find after a little of Algebra that the above nonlocal probability distribution is non-signalling upto $q - p \leq 0.5$ (equality is in the sense'just less than').

Thus as far as maximum probability of success of the Hardy's and the Cabello's arguments are concerned, none has got any practical edge over the other in **GNLT**, whereas in quantum theory probability of success of Cabello's argument is more than that of Hardy's.

3. General non-signaling probabilities satisfying Hardy-type non-locality argument for three two-level systems

Consider a physical system consisting of three subsystems shared among three far apart parties Alice, Bob and Charlie, in the framework of a general probabilistic theory. Assume that Alice, Bob and Charlie can measure one of the two observables X_i, Y_i, where i stands for the 1st (*i.e.*, Alice), 2nd (*i.e.*, Bob), or 3rd (*i.e.*, Charlie) on their respective subsystems. The outcomes of each such measurements can be either up (U) or down (D). We now consider all the sixty four joint probabilities $\text{Prob}(R_1 = j, R'_2 = k, R''_3 = l)$, where $R, R', R'' \in \{X, Y\}$ and $j, k, l \in \{U, D\}$. For the sake of notational simplicity, we will denote X by 0 and Y by 1 and also U by 0 and D by 1. We can thus denote the above-mentioned joint probabilities as $\text{Prob}(i = s_1, j = s_2, k = s_3)$, where $i, j, k \in \{0, 1\}$ and $s_1, s_2, s_3 \in \{0, 1\}$. To make it more readable, we will denote the probability $\text{Prob}(i = s_1, j = s_2, k = s_3)$ by $p_{is_1js_2ks_3}$, where $is_1js_2ks_3$ is the binary representation of the number $32i + 16s_1 + 8j + 4s_2 + 2k + s_3$.

Apart from being the of the interval $[0, 1]$, these sixty four joint probabilities p_0, p_1, $\ldots$, p_{63} must satisfy the normalization conditions:

$$\begin{gathered}
p_0 + p_1 + p_4 + p_5 + p_{16} + p_{17} + p_{20} + p_{21} = 1,\ p_2 + p_3 + p_6 + p_7 + p_{18} + p_{19} + p_{22} + p_{23} = 1,\\
p_8 + p_9 + p_{12} + p_{13} + p_{24} + p_{25} + p_{28} + p_{29} = 1,\ p_{10} + p_{11} + p_{14} + p_{15} + p_{26} + p_{27} + p_{30} + p_{31} = 1,\\
p_{32} + p_{33} + p_{36} + p_{37} + p_{48} + p_{49} + p_{52} + p_{53} = 1,\ p_{34} + p_{35} + p_{38} + p_{39} + p_{50} + p_{51} + p_{54} + p_{55} = 1,\\
p_{40} + p_{41} + p_{44} + p_{45} + p_{56} + p_{57} + p_{60} + p_{61} = 1,\ p_{42} + p_{43} + p_{46} + p_{47} + p_{58} + p_{59} + p_{62} + p_{63} = 1.
\end{gathered} \tag{22}$$

The no-signalling condition gives:

$$p_0+p_1=p_2+p_3,\ p_4+p_5=p_6+p_7,\ p_8+p_9=p_{10}+p_{11},\ p_{12}+p_{13}=p_{14}+p_{15},$$
$$p_{16}+p_{17}=p_{18}+p_{19},\ p_{20}+p_{21}=p_{22}+p_{23},\ p_{24}+p_{25}=p_{26}+p_{27},\ p_{28}+p_{29}=p_{30}+p_{31},$$
$$p_{32}+p_{33}=p_{34}+p_{35},\ p_{36}+p_{37}=p_{38}+p_{39},\ p_{40}+p_{41}=p_{42}+p_{43},\ p_{44}+p_{45}=p_{46}+p_{47},$$
$$p_{48}+p_{49}=p_{50}+p_{51},\ p_{52}+p_{53}=p_{54}+p_{55}, p_{56}+p_{57}=p_{58}+p_{59},\ p_{60}+p_{61}=p_{62}+p_{63}; \tag{23}$$

$$p_0+p_4=p_8+p_{12},\ p_1+p_5=p_9+p_{13},\ p_2+p_6=p_{10}+p_{14},\ p_3+p_7=p_{11}+p_{15},$$
$$p_{16}+p_{20}=p_{24}+p_{28},\ p_{17}+p_{21}=p_{25}+p_{29},\ p_{18}+p_{22}=p_{26}+p_{30},\ p_{19}+p_{23}=p_{27}+p_{31},$$
$$p_{32}+p_{36}=p_{40}+p_{44},\ p_{33}+p_{37}=p_{41}+p_{45},\ p_{34}+p_{38}=p_{42}+p_{46},\ p_{35}+p_{39}=p_{43}+p_{47},$$
$$p_{48}+p_{52}=p_{56}+p_{60},\ p_{49}+p_{53}=p_{57}+p_{61},\ p_{50}+p_{54}=p_{58}+p_{62},\ p_{51}+p_{55}=p_{59}+p_{63}; \tag{24}$$

$$p_0+p_{16}=p_{32}+p_{48},\ p_1+p_{17}=p_{33}+p_{49},\ p_2+p_{18}=p_{34}+p_{50},\ p_3+p_{19}=p_{35}+p_{51},$$
$$p_4+p_{20}=p_{36}+p_{52},\ p_5+p_{21}=p_{37}+p_{53},\ p_6+p_{22}=p_{38}+p_{54},\ p_7+p_{23}=p_{39}+p_{55},$$
$$p_8+p_{24}=p_{40}+p_{56},\ p_9+p_{25}=p_{41}+p_{57},\ p_{10}+p_{26}=p_{42}+p_{58},\ p_{11}+p_{27}=p_{43}+p_{59},$$
$$p_{12}+p_{28}=p_{44}+p_{60},\ p_{13}+p_{29}=p_{45}+p_{61},\ p_{14}+p_{30}=p_{46}+p_{62},\ p_{15}+p_{31}=p_{47}+p_{63}. \tag{25}$$

Given below are the Hardy-type non-locality conditions, which are incompatible with the notion of local-realism, but which all genuinely entangled pure state of three-qubit satisfy [22] with the maximum probabilty of success 12.5% [21] .

$$\begin{aligned} &p_{32}=0,\ p_8=0,\\ &p_2=0,\ p_{63}=0,\ p_0>0. \end{aligned} \tag{26}$$

Maximizing p_0 subject to satisfying all the conditions given in equations (22), (23), (24), (25), (26), with the help of *Mathematica*, gives the solution as

$$p_0^{max}=\frac{1}{2}, \tag{27}$$

while rest of the sixty four probabilities are given by[1]

$$p_3=p_{12}=p_{15}=p_{17}=p_{18}=p_{29}=p_{30}=p_{33}=p_{35}=p_{45}=p_{47}=p_{48}=p_{50}=p_{60}=p_{62}=\frac{1}{2},$$

$$\text{and } p_i=0 \text{ for all } i\in(\{1,2,\ldots,63\}-\{3,12,15,17,18,29,30,33,35,45,47,48,50,60,62\}). \tag{28}$$

Thus the maximum probability of success of Hardy's argument for three two-level systems too is more in GNLT than in the quantum theory.

4. Conclusion

In conclusion, we have shown here that in a more general framework of GNLT, the maximum probability of success of Hardy's argument can be enhanced for both the three two-level systems and for two two-level systems. Interestingly, the maximum success probability for both type of systems attains a common value 0.5. The same is the maximum probability of success of Cabello's argument for two two-level system in this more general framework.

Quantum non-locality has attracted much attention since its discovery because it relates quantum mechanics with special relativity. Special relativity forbids sending physical information with a speed greater than that of the light in vacuum. This is reflected in the quantum mechanical joint probabilities appearing both in the violation of Bell's inequality as well as in the fulfillment of Hardy's non-locality conditions, although there is no direct

[1] This solution set is not unique, but maximum probabilty is 0.5, for each such set.

relevance of special relativity in the postulates of non-relativistic quantum mechanics. These quantum mechanical joint probabilities are not only non-local but also non-signaling. The non-local probabilities, coming out from quantum mechanical states can give rise to the maximum violation up to the amount $2\sqrt{2}$ of Bell's inequality, whereas there are non-quantum mechanical non-local joint probabilities which give rise to the maximal possible algebraic violation (namely, 4) of Bell's inequality, without violating the relativistic-causality [9]. Why quantum theory can not provide more than $2\sqrt{2}$ violation of the Bell's inequality? By exploiting the theoretical structure of quantum mechanics it has been shown that a violation greater than $2\sqrt{2}$ will result in signalling in quantum mechanics [24, 25, 26]. We have seen in this paper that the no-signaling constraint cannot restrict the maximum value of the non-zero probability appearing in the Hardy's (Cabello's) argument to 0.09(0.11) all by itself. In a generalized non-signalling theory this value can go up to 0.5. It will be an interesting open question to find what feature of quantum mechanics along with no-signalling condition restricts these to their quantum values.

Acknowledgments
It is pleasure to thank Sibasish Ghosh, Guruprasd Kar, Samir Kunkri, Ramij Rahaman, Anirban Roy and Thomas Konrad for many useful discussions and suggestions.

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