

A theoretical derivation of discrete planetary distances from the sun

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Abstract. A theory of gravitation has previously been derived, based on a classical Lagrangian, which satisfies the classical tests for a theory of gravitation. When the Wilson-Sommerfeld quantisation rule is applied to the conservation equations of the theory, one finds discrete values for the planetary distances from the sun. The values for all planets agree with observation.

1. Introduction.

A series formula for predicting the distances of planets from the sun goes back to 1715 when it was mentioned by David Gregory in his *The Elements of Astronomy*. The classical formula, now known as the Bode-Titius formula, can be represented as

$$a_n = 4 + 3 \times 2^n, n = -\infty, 0, 1, 2 \dots \quad (1)$$

Earth: $n = 1$, $a_{\oplus} = 10$.

In 1764 Charles Bonnet referred to such a formula in his *Contemplation de la Nature*, to which Johann Daniel Titius added an unattributed comment, later removed to a footnote.

During 1772 Johann Elert Bode added the formula as a comment in the second edition of his *Anleitung zur Kenntniss des gestirnten Himmels*, but credited it to Titius in later versions.

The discovery of Uranus in 1781 gave credibility to the formula, but the discovery of Pluto in 1930, showing an error of 95.75%, seriously discredited the formula. The discovery of the Kuiper belt in 1992 further discredited the formula. Pluto, reclassified as a ‘dwarf planet’ is now regarded as the largest known member of the Kuiper belt. The formula has become a generic term for series formulas predicting planetary distances from the sun, but is now generally regarded as a fortuitous statistical correspondence. Nevertheless, the recent discovery that some of the exoplanets fit a comparable formula has renewed efforts to find a general formula for predicting planetary distances. Noteworthy are the works of Agnese[1, 2, 3], Neto[4], Giné[5], Christianto[6, 7, 8], Ilyanok[9], Chechelnitsky[10], Nottale[11, 12, 13, 14] and the report by the HARPS group[15]. However, none of the derivations of these groups proceed from an explicit theory of gravitation.

A comprehensive survey of the Bode-Titius rule up to 1972 is given by Nieto.[16]

The present paper is based on a theory of gravitation defined by a Lagrangian,

$$\mathcal{L} = -m_0(c^2 + v^2) \exp(R/r), \quad R = \frac{2GM}{c^2} = \text{Schwarzschild radius}, \quad (2)$$

where m_0 is the mass of a test particle moving with speed v about a central body of mass M .

This Lagrangian leads to equations of motion, which satisfy all the classical tests for a theory of gravitation. [17, 18]

The outline of this paper is as follows:

- (i) Start with the above Lagrangian.
- (ii) Derive the conservation equations for the energy E , total angular momentum $\mathbf{L}$ and the z -component of the total angular momentum L_z .
- (iii) Apply the Wilson-Sommerfeld quantization rule to the conserved quantities.
- (iv) Obtain the quantised values of E , $\mathbf{L}$ and L_z .
- (v) The average planetary distance $\bar{a}$ from the sun is related to the total energy of the planet by $\bar{a} = R_\odot/2(E - 1)$; $R_\odot \sim$ Schwarzschild radius.[19]
- (vi) The quantized planetary distances are derived and plotted.

2. The conservation equations.

Applying Hamilton's canonical equations to the Lagrangian gives

$$E = m_0 c^2 \frac{e^{R/r}}{\gamma^2} = \text{total energy} = \text{constant}, \quad (3)$$

$$\mathbf{L} = e^{R/r} \mathbf{M}, \quad |\mathbf{M}| = m_0 r^2 \frac{d\Psi}{dt}, \quad (4)$$

$$= \text{total angular momentum} = \text{constant}, \quad (5)$$

$$L_z = e^{R/r} m_0 r^2 \sin^2 \theta \dot{\phi}, \quad (6)$$

$$= z \text{ component of } \mathbf{L} = \text{constant},$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and the angle Ψ is measured in three dimensions.

The above conservation equations yield a general Kepler orbit:

$$\frac{d\Psi}{du} = (au^2 + bu + c)^{-1/2}, \quad u = 1/r, \quad (7)$$

$$u = K(1 + \epsilon \cos k\Psi). \quad (8)$$

- For an ellipse ($\epsilon < 1$) the value of $k \neq 1$ gives a precession of the ellipse, $\Delta\phi = 3\pi R/\bar{a}(1 - \epsilon^2)$.
- For a hyperbole ($\epsilon > 1$) one gets a deflection $2R/r_0$, where r_0 is the impact parameter of the curve.

The above predictions agree with observation.

3. Applying the Wilson-Sommerfeld rule.

The Wilson-Sommerfeld rule is well-known from the Old Quantum Theory:

$$\oint p_i dq_i = n_i w, \quad n_i = 0, 1, 2, \dots, \quad (9)$$

where w = an arbitrary constant, $p_i = \partial L / \partial q_i$ = conjugate momentum, q_i = position coordinate = r, θ, ϕ .

Applying the rule to the above conserved quantities gives

$$L_z = p_\phi = n_\phi \bar{w} = m \bar{w}, \quad m = 0, 1, 2, \dots, \quad (10)$$

$$L = p_\theta = [(n_\theta + n_\phi)/2] \bar{w} = k \bar{w}/2, \quad k = (n_\theta + n_\phi) = 0, 1, 2, \dots, \quad (11)$$

$$\oint p_r dr = \oint 2 \left[e^{2R/r} - E e^{R/r} - \frac{k^2 \bar{w}^2}{4r^2} \right]^{1/2} dr = n_r w. \quad n_r = 0, 1, 2, \dots, \quad (12)$$

where $\bar{w} = w/2\pi$ and we apply the convention $m_0 = c = 1$.

4. Deriving quantised E to first order in R/r .

From (12):

$$\begin{aligned}
n_r w &\approx 2 \oint \left[1 + \frac{2R}{r} - E \left(1 + \frac{R}{r} \right) - \frac{L^2}{r^2} \right]^{1/2} dr, \\
&= 2 \oint \left[\frac{R}{r}(2 - E) + (1 - E) - \frac{L^2}{r^2} \right]^{\frac{1}{2}} dr, \\
&\equiv 2 \oint \left[-A + \frac{2B}{r} - \frac{C}{r^2} \right]^{\frac{1}{2}} dr,
\end{aligned} \tag{13}$$

where $A = E - 1$, $B = R(2 - E)/2$, $C = L^2 = k^2 \bar{w}^2/4$.

Applying contour integration to (13):

$$\begin{aligned}
n_r \bar{w} &= 2 \left[-k\bar{w}/2 + \frac{R(2 - E)}{2\sqrt{E - 1}} \right], \\
\therefore E &= 2 \left(1 + \frac{n^2 \bar{w}^2}{4R^2} \right) \pm 2 \frac{n\bar{w}}{2R} \sqrt{1 + \frac{n^2 \bar{w}^2}{4R^2}}, \\
&= 2(1 + n^2 P^2) \pm 2nP\sqrt{1 + n^2 P^2},
\end{aligned} \tag{14}$$

where $P = \bar{w}/2R$ and $n = k + n_r = 0, 1, 2, \dots$ P is unique for a particular central body.

5. Determination of planetary distances.

A planet's average distance from the sun is given[19] by

$$\bar{a} = K/2(E - 1); \quad K \sim \text{Schwarzschild radius.} \tag{15}$$

Substituting for E from (14) and taking the negative sign gives

$$a_n = K/2(E - 1) = \frac{K/2}{2(1 + n^2 P^2) - 2nP\sqrt{1 + n^2 P^2} - 1}. \tag{16}$$

The constants K and P are unknowns to be determined from observation. K is a scale factor and is adjusted to best fit the observed distances after a value for P has been found. P is determined by assigning a distance $a_n = 10$ for $n = 5$. This value of n was found by trial and error after finding that no planets exist at $n = 1, 2$. This is shown in Table 1 below.

A good fit is found for $P = 200.00$ and $K = 0.000005$. The results for $n = 1$ to 16 are given in Table 1 below.

We note the following:

- Gaps appear at certain values of n . These gaps are also found by other authors,[1, 20, 21] but a number of systems of exoplanets do have planets at those values.[3, 14, 15]
- For $n > 15$ the values of E , and consequently also of a_n , are the same to nine significant digits. To determine distances for $n > 15$ we have to extrapolate to beyond $n > 15$. Drawing up a difference table of the ratios of the predicted distances a_n shows that the third differences are all zero to two decimals. This indicates that the a_n values can be represented by a parabola, $f(n) = a_0 + a_1 n + a_2 n^2$.

Table 1. Energy and planetary distances

n	planet	$E \times 10^8$	observed	a_n	$f(n)$	% relative difference
1		100000625	–	0.400		
2		100000156	–	1.600		
3	Mercury	100000069	3.87	3.600		6.99
4	Venus	100000039	7.23	6.399		11.49
5	Earth	100000025	10.00	10.007		0.07
6	Mars	100000017	15.20	14.393		5.31
7	Hungarias	100000013	20.99	19.523		6.31
8	Asteroid	100000010	27.70	25.811		7.71
9	Camilla	100000008	31.50	32.341		2.67
10		100000006	–	40.065		
11	Jupiter	100000005	52.00	47.935		0.73
12		100000004	–	58.356		
13		100000004	–	67.109		
14		100000003	–	78.952		
15	Saturn	100000003	95.40	95.870		0.49
16		100000003	–	95.870		
21	Uranus		191.90		185.57	3.30
27	Neptune		301.00		310.28	3.08
31	Pluto		395.00		411.22	4.11

We find a parabolic fit for $n = 1$ to 15,

$$f(n) = 0.4447n^2 - 0.5595n + 1.2028, \quad (17)$$

and apply this to $n > 15$. The extrapolated results for Uranus, Neptune and Pluto are indicated in Table 1. The respective plots are indicated in Fig. 1

Applying analogies with the Bohr atom, Agnese[1] finds a parabolic fit, $f(n) = 0.439n^2$. Christianto,[21, p124] applying a Cantorian Superfluid Vortex model, obtains $f(n) = 0.424n^2$. Values are adjusted for $a_5(\text{earth}) = 10$. None of these derivations are explicitly based on a theory of gravitation.

- The average percentage relative difference (APRD) for the first nine planets is 5.40%. The value of E is extremely sensitive to changes in the value of P . The smallest APRD of 4.64% is found for $P = 199.9900$; any changes to the third decimal increases the APRD considerably.

Venus presents an anomaly, which could be ascribed to effects during the early formation of the solar system. Without Venus and for $P = 199.99$ the APRD changes to 3.36%.

The APRD for the last three planets is 3.50%.

- By analogy the results of Agnese, Christianto and Nottale also apply to the theory of this paper, e.g. calculations of the distances of the satellites of the planets, of the exoplanets, the planetary systems of pulsars and the rings of Saturn[1]. Agnese[1] also finds discrete values for the angular momentum ('spin') of celestial bodies.

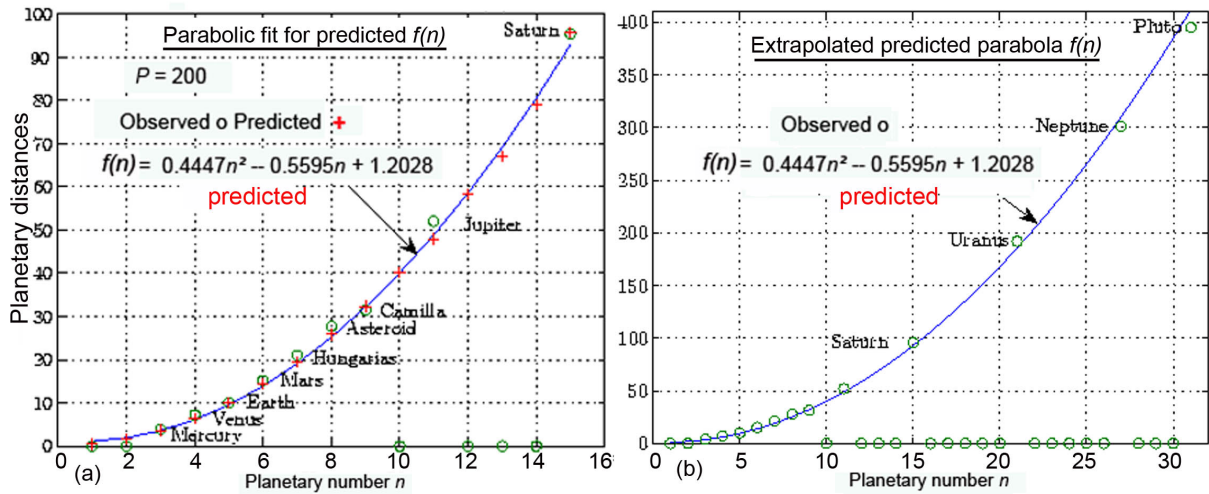


Figure 1. Planetary distances: (a) For $1 < n < 15$. (b) Extrapolated for $1 < n < 30$.

6. Conclusion

Our prediction of discrete values for planetary distances from the sun is the first to proceed entirely from a theory of gravitation. The application to exoplanets and to the satellites of the planets (in preparation) shows a general applicability of the constant w as a universal gravitational constant, analogous to Planck's constant in microphysics.

7. References

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