On the Shape of Rotating Stars

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Abstract. In this paper, I adapt a recent work by Zahn et al. on the shape of rapidly rotating stars to explore the degree of oblateness for uniform and differential rotation of the star. I also discuss the relation of these results with the classical Roche limit.

1. Introduction

The shape assumed by a rapidly rotating star, has been the subject of investigation for many years. Recent advances in long baseline interferometry, now make it possible to measure the degree of flattening of some nearby stars. Domiciano de Souza et al.[1] recently announced that the Be star Achernar is oblate in shape with ratio of the major to minor axes of 1.56. This unexpectedly high value raises challenging problems for current models. An early model that predicts the shape of a star rotating at critical velocity is due to Roche. In this model the entire mass of the star is treated as if it were concentrated at its centre. The maximum flattening ratio predicted by this model is 1.5. Recently Zahn et al.[2] attempted to predict this measured value for the flattening of Achernar by modifying the Roche model. In their model the mass is no longer concentrated in the centre of the star but is now contained in an oblate spheroid. This adds a quadrupolar term to the Roche theory. They illustrated the impact of inclusion of the quadrupolar moment by considering a 7 Solar mass star in several different stages of evolution and undergoing three forms of rotation: uniform, differential and shellular. They were able to obtain flattening ratios of greater than 1.5.

In this paper we explore the effects of the inclusion of the quadrupolar moment in the simpler case of a polytrope and examine the flattening ratio for different polytropic indices and for angular velocities ranging from zero to critical. We calculate the quadrupolar moment of the rotating polytrope by a linear perturbation method first described by Sweet [3], and speculate on the inclusion of higher order moments.

2. Polytropes

A polytrope is a star in which the pressure is proportional to a power of the density[4],

$$P = K\rho^{\gamma'} \tag{1}$$

where K is a constant, and γ' is called the polytropic power. The equation governing the structure of a non-rotating polytrope is the Lane-Emden equation, given by

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \tag{2}$$

where θ and ξ are dimensionless variables related to the density and the radius respectively. The boundary conditions for equation (2) are $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. The radius of the star is given by

$$R = \left[\frac{(n+1)K}{4\pi G}\lambda^{(1/n)-1}\right]^{1/2} \xi_1 \tag{3}$$

where ξ_1 is the first zero of the solution to equation 2, and λ is the central density of the polytrope. The Mass of the star is given by

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G}\lambda^{(1/n)-1}\right]^{3/2} \xi^2 \, \frac{d\theta}{d\xi} \tag{4}$$

evaluated at $\xi = \xi_1$.

3. Quadrupolar moment

3.1. Flattening of the star

Consider a non-rotating star of mass M and radius R. The stellar surface can be taken to coincide with a level surface of suitably chosen low pressure [2]. Since a polytrope is a barotropic system, isobaric surfaces coincide with isopycnic surfaces, and also with level surfaces of the effective gravitational potential. This potential, for a star of radius R rotating at constant angular velocity Ω is given by

$$\Phi_{\text{eff}} = -\frac{GM_r}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2(\theta) \tag{5}$$

where θ is the colatitude. To determine the value Φ_{eff} on the surface equipotential, note that at the pole centrifugal force is zero and $r = R_P$, so that at the surface $\Phi_{\text{eff}} = -GM/R_P$. At the stellar equator, therefore, we have

$$\frac{GM}{R_E} + \frac{1}{2}\Omega^2 R_E^2 = \frac{GM}{R_P} \tag{6}$$

or

$$1 + \frac{1}{2}\frac{\Omega^2 R_E^3}{GM} = \frac{R_E}{R_P} \tag{7}$$

Equation (7) can be used to calculate the ratio of equatorial to polar radii for given angular velocity. The critical angular velocity is defined to be that angular velocity at which the star must rotate in order for the effective gravity at a point on its surface at its equator to be zero. This gives,

$$\Omega_{\text{crit}}^2 = \frac{GM}{R_E^3} \tag{8}$$

3.2. The Quadrupolar correction

In the Roche Model, it is assumed that the gravitational potential of the star is the same as that which would be set up were its entire mass concentrated at its centre. This is a good first approximation. However, rotation causes redistribution of the stellar mass making the star oblate and causing the mass distribution to deviate from spherical symmetry. The gravitational potential outside the star may, therefore, be expanded as a multipole series of the form

$$\phi(r,\theta) = -\frac{GM}{r}\left[1 - \sum_{l=2}^{\infty}\left(\frac{R_0}{r}\right)^l J_l P_l(\cos(\theta))\right] \tag{9}$$

where R_0 is the radius of the spherically symmetric reference model, J_l is a dimensionless constant that measures the degree of oblateness, $P_l(\cos\theta)$ is the Legendre polynomial of degree l. Because we have assumed symmetry about the equatorial plane, only the even Legendre polynomials are required in the expansion.

3.3. First Order Perturbation of the Gravitational Potential

We use a generalised first order perturbation method, first described by Sweet [3], to derive an expression for the distortion of the equipotential surfaces by rotation. The equations of hydrostatic equilibrium in the rotating frame, assuming azimuthal symmetry, are given by

$$\frac{\partial P'}{\partial r} = \rho'\frac{\partial \phi'}{\partial r} + \rho' f_r \tag{10}$$

$$\frac{\partial P'}{\partial \theta} = \rho'\frac{\partial \phi'}{\partial \theta} + \rho' f_\theta \tag{11}$$

where $\rho' f_r$ and $\rho' f_\theta$ are the components of the centrifugal force, ϕ' is Sweets gravitational potential (which is the negative of the potential ordinarily used), P' is the hydrostatic pressure, and ρ' is the density of the material. Dashes are used here to denote values in the perturbed star, while undashed quantities denote values in the unperturbed star: that is

$$P' = P + \delta P \tag{12}$$

$$\rho' = \rho + \delta\rho \tag{13}$$

$$\phi' = \phi + \delta\phi \tag{14}$$

On eliminating P' from equations (11) one obtains, correct to the first order,

$$\frac{\partial \rho'}{\partial \theta} = -\frac{\chi}{g}\frac{d\rho}{dr} + \frac{1}{g}\left(\rho\frac{\partial f_r}{\partial \theta} - \frac{\partial}{\partial r}(r\rho f_\theta)\right) \tag{15}$$

where g is the local gravitational acceleration and $\chi = \partial\phi'/\partial\theta$. Using Poisson's equation to substitute for $\partial\rho'/\partial\theta$ in equation (15)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\chi\sin\theta\right]\right) = \frac{4\pi G}{g}\frac{d\rho}{dr}\chi + \frac{4\pi G}{g}\left[\frac{\partial}{\partial r}\left(r\rho f_\theta - \rho\frac{\partial f_r}{\partial\theta}\right)\right] \tag{16}$$

The form of the left hand side of equation (16) suggests that we expand χ in terms of spherical functions. Expanding the centrifugal acceleration, and the gravitational potential χ in terms of Legendre polynomials one obtains

$$\left.\begin{array}{rcl} f_r &=& \Omega^2 r\sin^2\theta = \sum a_l(r)P_l(\cos\theta) \\ f_\theta &=& \Omega^2 r^2\sin\theta\cos\theta = -\sum b_l(r)\frac{d}{d\theta}P_l(\cos\theta) \\ \frac{d\phi}{d\theta} &=& \sum c_l\frac{d}{d\theta}P_l(\cos\theta) \end{array}\right\} \tag{17}$$

Now

$$a_l(r) = \alpha_l r\Omega^2 \tag{18}$$
$$b_l(r) = \beta_l r\Omega^2 \tag{19}$$

where α_l and β_l depend on l. Substituting equations (18) and (19) into equation (16), we have

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dc_l}{dr}\right) - \left[\frac{l(l+1)}{r^2} + \frac{4\pi G}{g}\frac{d\rho}{dr}\right]c_l = \frac{4\pi G}{g}\left[\frac{d}{dr}(r\rho b_l) + \rho a_l\right] \tag{20}$$

Equation (20) is known as the perturbed Poisson equation. To ensure regularity of the functions c_l at at the origin and at the surface of the star, the boundary conditions required for $l \geq 1$ are [5],

$$c_l(0) = 0 \tag{21}$$
$$(l+1)c_l(R) = -\frac{dc_l}{dr}(R) \tag{22}$$

3.4. Re-scaled Perturbed Poisson Equation
To simplify integration of equation (20), we rescale it. Define $x = r/R_0$, $h(x) = \Omega^2(x)/\Omega_s^2$ and $\phi_l = c_l/\Omega_s^2 R_0^2$. Then, using equations (18) and (19), equation (20) becomes

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\phi_l}{dx}\right) - \left[\frac{l(l+1)}{x^2} + \frac{4\pi G R_0}{g}\frac{d\rho}{dx}\right]\phi_l = \frac{4\pi G R_0}{g}\left[\frac{d}{dx}(x^2\rho\beta_l h(x)) + \rho\alpha_l h(x)\right] \tag{23}$$

The domain of integration is $x = [0, 1]$. One can now calculate the moments J_l from the values of ϕ_l at the surface,

$$J_l = \left(\frac{\Omega_s^2 R_0^3}{GM}\right)\phi_l(1) \tag{24}$$

3.5. Quadrupolar Moment
We are interested only in J_2, the first correction to the gravitational potential. For $l = 2$, we get $\alpha_2 = -2/3$ and $\beta_2 = 1/3$. Equation (23) thus becomes

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\phi_2}{dx}\right) - \left[\frac{6}{x^2} + \frac{4\pi G R_0}{g}\frac{d\rho}{dx}\right]\phi_2 = \frac{4\pi G R_0}{3g}x^2\frac{d}{dx}(\rho\ h(x)) \tag{25}$$

and equation (24) yields

$$J_2 = \left(\frac{\Omega_s^2 R_0^3}{GM}\right)\phi_2(1) \tag{26}$$

3.6. Quadrupolar Correction to the Flattening of the Star
Inserting the quadrupolar correction into equation (7), and substituting expression (26) for J_2, we obtain

$$1 + \frac{1}{2}(1 + \phi_2(1))\frac{\Omega^2 R_E^3}{GM} = \frac{R_E}{R_P} \tag{27}$$

4. Results

We solved equation (2) for various values of n, to obtain ρ, $d\rho/dx$, M and R_0. For the central density, we used the value 1×10^{15} kg.m^{-3}, and took the value of K to be 2.76×10^9 in the appropriate units. We then used these quantities to solve the perturbed Poisson equation (25) and thus calculate the quadrupolar moment of the associated perturbed gravitational potential. We used the rotation profile given by

$$h(x) = \frac{1+a}{1+ax^2} \tag{28}$$

The value $a = 0$, corresponds to solid body rotation, while non zero values yield differential rotation. We calculated the cases corresponding to values $a = 0, 1, 2, 3, 4$. Figure 1 shows a plot of the flattening ratio as a function of Ω/Ω_c for a polytrope with index $n = 1$. The lowest curve on the figure, corresponds to $a = 0$, the next one up corresponds to $a = 1$ and so on. This is the general shape of the curves for different differential rotation profiles. Figure 2 shows an enlargement of the region for the range $0.8 \leq \Omega/\Omega_c \leq 1$, where the flattening ratio reaches its largest values. The panels in figure 2, represent respectively polytropes of indices $n = 1, 2, 3, 4$. As can be seen from the plots in figure 2, the separation of the curves for increasing a decreases with increasing index n. Note also that greatest degree of flattening occurs for the lowest polytropic index. This can be attributed to the fact that the polytropes of lower index are less centrally condensed.

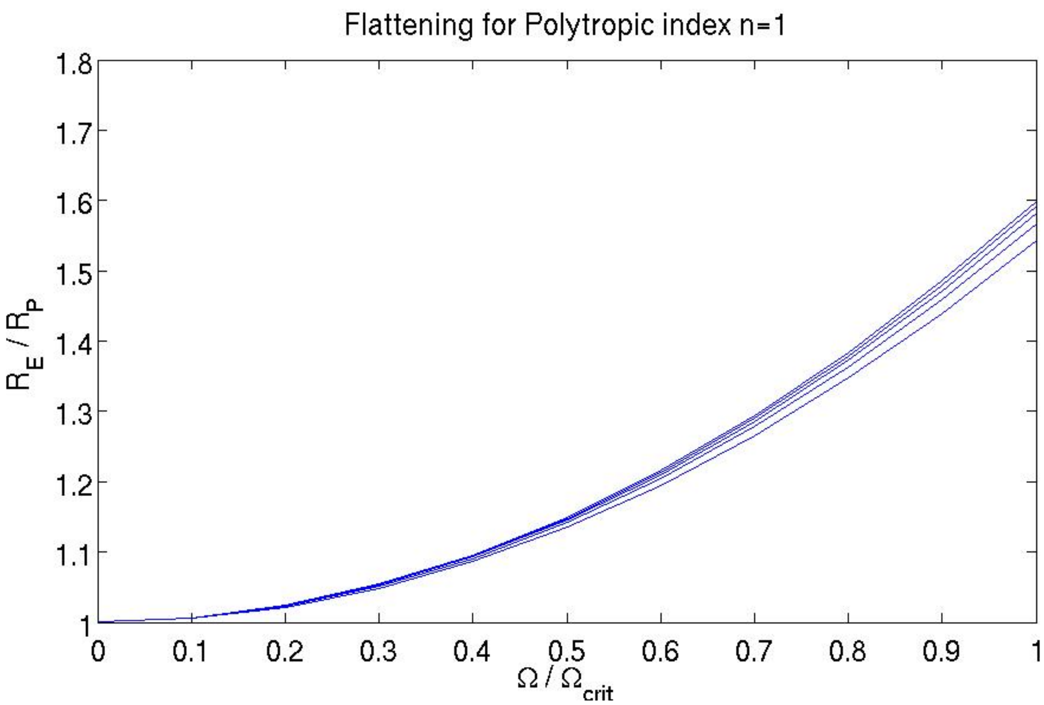

Figure 1. Flattening of polytrope with index n=1.

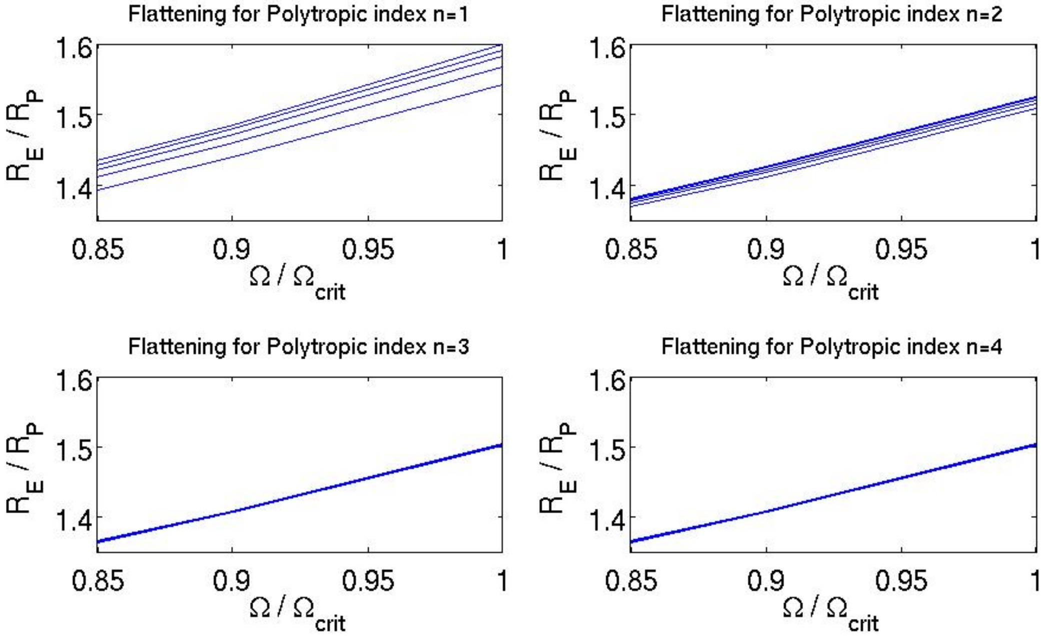

Figure 2. Flattening of polytropes with indices from n=1 to n=4, from left to right, top to bottom.

5. References

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