Representation of the quantum and classical states of light carrying orbital angular momentum

Humairah Bassa and Thomas Konrad

Quantum Research Group, University of KwaZulu-Natal, Durban 4001, South Africa

E-mail: `hbassa786@gmail.com`

Abstract. Ever since the discovery, by Allen *et al.* that beams which carry a quantized orbital angular momentum could be easily realized in the laboratory, there have been many classical and quantum optical experiments that utilize this new optical degree of freedom. It is therefore necessary to familiarize oneself with the representation of the quantum and classical states of light in order to explain the results of these experiments. Classical paraxial light is described by a superposition of analytical solutions to the paraxial wave equation. However, due to the well-known analogy between quantum mechanics and paraxial optics we can apply operator algebra to the classical states of a light beam. A short review of the applications of this formalism is given. For the quantum-mechanical description of light, we quantize the paraxial light beams and this leads to a description of the photon in terms of creation operators acting on the vacuum state. The analogy between the quantum and classical states of light is analyzed.

1. Introduction

The laws which govern the behaviour of electromagnetic radiation, and more specifically light, were first compiled into a self-consistent theory by James Maxwell in 1864 and since then the four fundamental equations of electromagnetism have been known as Maxwell's equations. It is well-known from Maxwell's theory that light carries both energy and an associated momentum, which consists of linear and angular contributions. In 1909, Poynting proved that circularly polarized light carries spin angular momentum, which is related to the polarization of the light beam. This was experimentally verified by Beth, 20 years later, using a quarter-wave plate suspended from a fine quartz fibre. Although the concept of a light beam possessing orbital angular momentum is not new, it was only in 1992 that Allen *et al.* [1] realized that it was possible to easily produce a light beam possessing a well-defined orbital angular momentum in the laboratory. In their seminal work they showed that within the paraxial approximation, any monochromatic beam with an azimuthal phase dependence of $\exp(il\phi)$ will possess an orbital angular momentum of $l\hbar$ per photon. Since then there have been many experiments which utilize this new degree optical degree of freedom in novel experiments of quantum and classical optics.

It is therefore necessary to be able to represent the quantum and classical states of light in order to describe the results of experiments. This article reviews the use of operator algebra or Dirac notation for paraxial light beams. This technique is advantageous because it allows one to calculate quantities associated with the field in a very simple way as well as to analyze the behaviour of light beams during propagation without going into detailed calculations. We then quantize the field of a paraxial light beam in order to obtain a description of a single-photon state in terms of creation operators acting on the vacuum state. Finally, we explain the analogy between the representation of the classical and quantum states of light.

2. Operator algebra for laser modes

Maxwell's equations can be combined to give scalar wave equations for each component of the electric, **E** and magnetic, **B** fields of a light beam of the form,

$$\left(\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = 0. \tag{1}$$

Our aim is to describe a laser beam which is a collimated, monochromatic light beam whose transverse beam profile, $u(x, y, z)$, varies slowly with longitudinal propagation in the z direction. In this case we use the separation of variables $\Psi(x, y, z, t) = u(x, y, z)\exp[i(kz - \omega t)]$ and invoke the paraxial approximation (where neglect $\frac{\partial^2}{\partial z^2}$ in favour of $k\frac{\partial}{\partial z}$) which leads to the paraxial wave equation

$$2ik\frac{\partial}{\partial z}u(x, y, z) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x, y, z). \tag{2}$$

We notice two properties of this equation that gave Stoler [2] a strong motivation to apply the operator formalism (which is used quantum-mechanically to describe the time-evolution of the quantum-mechanical states) to describe the longitudinal propagation of classical light beams. Firstly, the form of the paraxial wave equation (Eq. (2)) is analogous to the two-dimensional time-independent Shrödinger equation for a free particle with the z co-ordinate replacing the time variable, t and the wavelength replacing Planck's constant. Secondly, Eq. (2) has the explicit solution

$$u(x, y, z) = \exp\left[\frac{i(z - z_0)}{2k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]u(x, y, z_0), \tag{3}$$

which tells us that if we know the transverse profile of the beam at a point z_0 we can use the exponential factor to find the transverse profile of the beam at a later position z. This means that this exponential factor is analogous to the time-evolution operator that describes the time displacement of the wave function of a particle from time t_0 to t. We can therefore transform Eq. (3) into an operator relation in Dirac notation as follows.

In this notation we can represent the transverse beam profile $u(x, y, z)$ as a state vector $|u(z)\rangle$. Such a state vector lies in the Hilbert space $L^2(\mathbb{C}^2)$ and is normalized, obeying the identity

$$\langle u(z)|u(z)\rangle \equiv \int d\mathbf{r}u^*(\mathbf{r}, z)u(\mathbf{r}, z) = 1 \tag{4}$$

where $\mathbf{r} = (x, y)^T$ is the position in the transverse plane. As well as showing the normalization of the state vectors this equation shows how the inner product is defined on this space. When u is normalized for one position z it is automatically normalized for all other z values [5]. The basis vectors in this space are $|\mathbf{r}\rangle$, which are eigenvectors of the transverse position operator $\hat{\mathbf{r}} = (\hat{\mathbf{x}}, \hat{\mathbf{y}})$ that acts on state vectors in this space. If we take the inner product of the state vector $|u(z)\rangle$ with a basis vector we recover the beam profile,

$$u(x, y, z) = \langle\mathbf{r}|u(z)\rangle. \tag{5}$$

The differential operators $-i\frac{\partial}{\partial x}$ and $-i\frac{\partial}{\partial y}$ which act in the function space containing $u(x, y, z)$ as shown in Eq. (3) are represented by the momentum operators $\hat{\mathbf{p}}_x$ and $\hat{\mathbf{p}}_y$, respectively, in this Hilbert space. The commutation relations between the position and momentum operators are defined as in quantum mechanics. With these preliminaries, we may now express Eq. (3) in Dirac notation as

$$|u(z)\rangle = \hat{\mathbf{U}}|u(z_0)\rangle, \tag{6}$$

where the operator $\hat{\mathbf{U}}$, which governs the propagation of the state from position z_0 to z, is given by,

$$U = \exp\left[-\frac{i(z-z_0)}{2k}\hat{\mathbf{p}}^2\right] \tag{7}$$

with $\hat{\mathbf{p}}^2 = \hat{\mathbf{p}}_x^2 + \hat{\mathbf{p}}_y^2$. Please note that even though we are using the operator algebra, which is usually used in quantum mechanics, we are still in the purely classical regime.

The expectation value of an operator, $\hat{\mathbf{O}}$ is defined in the standard way as

$$\langle O\rangle = \langle u(z)|\hat{\mathbf{O}}|u(z)\rangle = \int d\mathbf{r}u^*(\mathbf{r},z)\hat{\mathbf{O}}u(\mathbf{r},z). \tag{8}$$

In this equation the states are z-dependent and the operators "stationary", which is the Schrödinger picture. We now move to the Heisenberg picture where the operators become z-dependent, so that the evolution of the expectation value of an operator can be expressed as,

$$\langle O\rangle = \langle u(z_0)|\hat{\mathbf{U}}\hat{\mathbf{O}}\hat{\mathbf{U}}|u(z_0)\rangle. \tag{9}$$

A complete an orthogonal set of solutions to the paraxial wave equation (Eq. (2)) is given by the Laguerre-Gaussian (LG) modes, u_{pl}. This means that the set of LG modes forms a basis for paraxial light beams, so any laser beam can be expanded as a superposition of LG modes. It is well known that the analytic form of these modes resemble the wave functions of the stationary states of a two-dimensional quantum-mechanical harmonic oscillator. Therefore Nienhuis and Allen [5] were able to use operator algebra to obtain ladder operators of the harmonic oscillator which allow one to generate higher-order LG modes from the fundamental Gaussian mode.

Let us consider an example given in [3] that shows how this operator formalism simplifies the calculations of quantities associated with the electric field. The full scalar version electric field which includes time dependence on it can be written as

$$\langle\mathbf{r}|E\rangle = E_0\exp[i(kz-wt)]\langle\mathbf{r}\mid u\rangle \tag{10}$$

where E_0 is the amplitude of the field and $u \equiv u(z)$ for convenience. If $u(\rho,\phi,z) = u_0(\rho,z)e^{il\phi}$, which is a Laguerre-Gaussian mode in cylindrical coordinates, then the expectation value of the angular momentum operator is

$$\langle L_z\rangle = \left\langle E\left|i\hbar\frac{\partial}{\partial\phi}\right|E\right\rangle = |E_0|^2\hbar l\langle u|u\rangle. \tag{11}$$

The linear momentum of the state in the direction of propagation, z is given by

$$\langle p_z\rangle = \left\langle E\left|i\hbar\frac{\partial}{\partial z}\right|E\right\rangle = |E_0|^2\hbar k\langle u|u\rangle, \tag{12}$$

while the energy of the state is

$$\langle W\rangle = \int\int dxdyE^*(\mathbf{r},t)\left(i\hbar\frac{\partial}{\partial t}\right)E(\mathbf{r},t) = |E_0|^2\hbar w\langle u|u\rangle. \tag{13}$$

Therefore the ratio of the orbital angular momentum carried by the wave to the energy that it carries is

$$\frac{\langle L_z\rangle}{\langle W\rangle} = \frac{l}{\omega} \tag{14}$$

while the ratio of orbital angular momentum to linear momentum is

$$\frac{\langle L_z \rangle}{\langle p_z \rangle} = \frac{l}{k}. \tag{15}$$

These relations are valid classically as well as quantum mechanically, since $\hbar$ does not appear in it. Every photon has an energy of $\hbar\omega$ and a linear momentum of $\hbar k$ therefore for the ratios to be valid on a single-photon level, each photon must possess a well-defined an orbital angular momentum of $l\hbar$. Thus we have, in a simple way, arrived at the conclusion reached by Allen *et al.* [1], that every photon in a Laguerre-Gaussian beam carries a quantized orbital angular momentum of $l\hbar$ per photon. Also note that due to Eq. (9) the quantities in Eq.s (11), (12) and (13) are all conserved during free space propagation since the operators in these equations commute with $\hat{\mathbf{p}}^2$ of the propagation operator $\hat{\mathbf{U}}$ [4].

This operator formalism that we have developed is not limited to free-space propagation. Indeed many authors [2, 4, 5] have applied it to the propagation of beams through thin lenses. We can implement the effect of this lens with a focal length f by means of an operator $\hat{\mathbf{T}}_{lens}$ that is is a function of the transverse coordinate operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ [2]

$$\hat{\mathbf{T}}_{lens} = \exp\left[-\frac{ik}{2f}(\hat{\mathbf{x}}^2 + \hat{\mathbf{y}}^2)\right]. \tag{16}$$

Let us consider a system which consists of two identical cylindrical lenses orientated along the x-axis with focal length f and separated from each other by a distance $2d$ with free propagation between the lenses. The propagation operator for this system is given by

$$\hat{\mathbf{U}}_{cyl} = \hat{\mathbf{T}}_{lens}\hat{\mathbf{U}}\hat{\mathbf{T}}_{lens} = \exp\left[\frac{-ik\hat{\mathbf{x}}^2}{2f}\right]\exp\left[\frac{-d\hat{\mathbf{p}}^2}{k}\right]\exp\left[\frac{-ik\hat{\mathbf{x}}^2}{2f}\right] \tag{17}$$

Using this propagation operator van Enk and Nienhuis [4] were able to prove, in a simple way, that if $d = f/\sqrt{2}$ then this setup converts a Laguerre-Gaussian mode which contains orbital angular momentum, into a beam that contains no orbital angular momentum. This setup is known as an Astigmatic Mode Converter, and in [6], Allen *et al.* were able to arrive at this conclusion in a more complex way.

3. Quantum mechanical representation of classical light

The electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields can be written in terms of vector potentials and it is often easier to solve Maxwell's equations corresponding to these potentials rather than those given for the $\mathbf{E}$ and $\mathbf{B}$ fields. The first step for quantization of the electromagnetic field is to express this vector potential in terms of a basis. In the Coulomb gauge, the vector potential is usually written in the well-known continuous plane wave expansion. The second step is to convert the complex amplitudes that appear in the plane wave expansion into creation ($\hat{\mathbf{a}}_s^\dagger(\mathbf{k})$) and annihilation ($\hat{\mathbf{a}}_s(\mathbf{k})$) operators which obey the commutation relation $[a_s(\mathbf{k}), a_s'^\dagger(\mathbf{k}')] = \delta_{ss'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. The vector potential becomes a quantum mechanical operator given by

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_s \int d^3\mathbf{k}\left(\frac{\hbar}{16\pi^3\varepsilon_0 c|\mathbf{k}|}\right)^{\frac{1}{2}}[\boldsymbol{\epsilon}_s(\mathbf{k})\hat{\mathbf{a}}_s(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-c|\mathbf{k}|t)} + \text{h.c}]. \tag{18}$$

where h.c. refers to the hermitian conjugate of the first term.

This is a very general quantization of an electromagnetic field. We want to find a representation of field for laser beam and a novel method in which to obtain this is provided by Calvo *et al.* [7]. They introduce a dispersion relation which allows them to write the wave vector as $\mathbf{k} = q\mathbf{q} + k_0(1-\theta^2)\mathbf{u}_z$ where $\mathbf{q}$ is the transverse wave vector and $\theta = \frac{q}{\sqrt{2k_0^2}}$ is the parameter that governs the degree of paraxiality. By substituting this into the equation Eq. (18) and invoking the paraxial approximation (i.e. $\theta \ll 1$) they find,

$$A_P(\mathbf{r},t) = \sum_s \int_0^\infty dk_0 \left(\frac{\hbar}{16\pi^3\varepsilon_0 ck_0}\right)^{\frac{1}{2}} \int d^2\mathbf{q}[\boldsymbol{\epsilon_s}\hat{\mathbf{a}}_s(\mathbf{q},k_0)e^{ik_0(z-ct)}e^{i(\mathbf{q}\cdot r_\perp - k_0\theta^2 z)} + \text{h.c.}] \tag{19}$$

where they have retained only θ^2 which appears in the second phase factor is retained because it is the only relevant paraxial contribution. The transverse basis of Laguerre-Gaussian modes are a well known orthogonal and complete basis set for paraxial light beams so the vector potential can now be expressed in this basis. By noticing that the structure of the $\mathbf{q}$-integrand in (19) resembles the paraxial angular spectrum, the second phase factor can be written as,

$$e^{i[\mathbf{q}\cdot\mathbf{r}_\perp - k_0\theta^2 z]} = \sum_{l,p} \mathcal{LG}^*_{l,p}(\mathbf{q})LG_{l,p}(\mathbf{r}_\perp, z; k_0) \tag{20}$$

where $LG_{l,p}(\mathbf{r}_\perp, z; k_0)$ are the Laguerre-Gaussian modes and $\mathcal{LG}_{l,p}(\mathbf{q})$ are the Fourier transformed profiles at $z = 0$. The proof of this relation and the expressions for these modes can be found in the Appendix of [7]. Thus Eq. (19) can be written as

$$A_P(\mathbf{r},t) = \sum_{s,l,p} \int_0^\infty dk_0 \left(\frac{\hbar}{16\pi^3\varepsilon_0 ck_0}\right)^{\frac{1}{2}} [\boldsymbol{\epsilon_s}\hat{\mathbf{a}}_{s,l,p}(k_0)e^{ik_0(z-ct)}LG_{l,p}(\mathbf{r}_\perp, z; k_0) + \text{h.c.}] \tag{21}$$

where they introduce the Laguerre-Gaussian mode annihilation operators,

$$\hat{\mathbf{a}}_{s,l,p}(k_0) = \int d^2\mathbf{q}\mathcal{LG}^*_{l,p}(\mathbf{q})\hat{\mathbf{a}}_s(\mathbf{q},k_0), \tag{22}$$

which satisfy the commutation relation

$$[\hat{\mathbf{a}}_{s,l,p}(k_0), \hat{\mathbf{a}}^\dagger_{s',l',p'}(k_0')] = \delta_{ss'}\delta_{ll'}\delta_{pp'}\delta(k-k_0'). \tag{23}$$

The most general paraxial single-photon state in the Laguerre-Gaussian basis is then given by

$$|\psi\rangle = \sum_{s,l,p} \int_0^\infty dk_0 C_{s,l,p}(k_0)\hat{\mathbf{a}}^\dagger_{s,l,p}(k_0)|0\rangle \tag{24}$$

where $|0\rangle$ represents the vacuum state and the complex coefficients $C_{s,l,p}$ satisfy the normalization condition $\sum_{s,l,p}\int_0^\infty dk_0\, |C_{s,l,p}|^2 = 1$. This is a Fock state where the complex coefficients represent the probability of finding the photon in the state $|s,l,p,k_0\rangle$ with polarization s, wave vector k_0 along the z-axis corresponding to an LG mode with azimuthal mode index, l and radial mode index, p. The Fock state of a photon in a linearly polarized, monochromatic beam in the LG basis is given by

$$|\psi_{lp}\rangle = \sum_{l,p} f_{l,p}\hat{\mathbf{a}}^\dagger_{s,l,p}(k_0)|0\rangle, \tag{25}$$

where the complex coefficients $f_{l,p}$ once again satisfy the normalization condition $\sum_{l,p}|f_{l,p}|^2 = 1$ and represent the probability of finding the photon in the state $|l,p\rangle$. For experiments that utilize the orbital angular momentum of light beams and photons it is common to experimentally realize linearly polarized, monochromatic LG beams (with specific l and p values) in the laboratory so we should be able to represent the state of a photon in this mode. Such a state will contain a quantized orbital angular momentum of $l\hbar$ and can be written as

$$|lp\rangle = \hat{\mathbf{a}}_{s,l,p}(k_0)|0\rangle = \int d^2\mathbf{q}\mathcal{LG}_{pl}(\mathbf{q})\hat{\mathbf{a}}^\dagger_s(\mathbf{q},k_0)|0\rangle. \tag{26}$$

where we have used Eq. (22).

4. Discussion

We have shown how to use operator algebra in order to represent paraxial light beams and to analyze the propagation of these beams through free-space and thin lenses. The algebra simplified this kind analysis and the calculations of quantities associated with the light beam. We then presented the quantization of a paraxial light beam and the representation of a single-mode Fock state in the LG basis.

Although operator algebra is a technique used in quantum mechanics it is not specific to it; it is just an algebra for dealing with vectors. Since functions can be represented as vectors which form a Hilbert space it is perfectly acceptable to apply this algebra to the beam profiles of the electric field, as we have done. We must therefore note that by using this algebra we have not transitioned to the quantum level; we are still in the classical regime. Our state vectors represent the transverse beam profile of the electric field which can be measured; they do not represent the wave function of a photon in that mode of the electric field. There does however, exist an analogy between these two notations.

Consider the state in Eq. (25) which is a single-mode Fock state in the LG basis. This states represents the creation of a photon or a single excitation with a probability amplitude $f_{l,p}$ in one LG mode u_{pl} with a specific index p and l and a vacuum in all other modes (i.e. modes with different p and l values). This state can be written as $|\psi_{lp}\rangle = \int \langle \mathbf{r}|\psi_{lp}\rangle |\mathbf{r}\rangle d^2\mathbf{r}$, then $\psi(\mathbf{r})$ is which is the transverse wave function in position representation or the transverse spatial distribution of the single photon field [8]. We can now adopt the operator algebra where we denote the wave-function as the ket $|\psi_{lp}(\mathbf{r}, z)\rangle$ and use the operators we have discussed in Section 2 to propagate this wave-function in free-space and through thin lenses. We therefore notice that optical elements such as the mode converters that we have mentioned, also function analogously on a single single photon level.

Although there is an analogy between the two representations, the difference becomes clear during measurement. Note that a general paraxial light beam with state vector $|u(z)\rangle$ is a superposition of the LG mode functions, $|u_{pl}(z)\rangle$, since the LG modes form a basis for such beams (i.e. $|u(z)\rangle = \sum_{p,l} \alpha_{p,l} |u_{pl}(z)\rangle$). If $|\langle \mathbf{r}|u(z)\rangle|^2 = |u(x, y, z)|^2$ we obtain the intensity of the electric field, which is the sum of the intensities of each of the LG modes in the superposition. This is a measurable quantity. The wave-function on the other hand gives only the probabilities of finding a photon in a particular state. For example, if we perform the measurement of the state in Eq. (25) in the LG basis, the probability of finding a photon in the state $|lp\rangle$ is given by $|f_{l,p}|^2$. Thus we note that it is the measurement of a state that distinguishes its quantum or classical nature.

References

[1] Allen L, Beijersbergen M W, Spreeuw R J C and Woerdman J P 1992 *Phys. Rev.* A **45** 8185
[2] Stoler D 1981 *J. Opt. Soc. Am.* **71** 334
[3] Schechner Y Y and Shamir J 1996 *J. Opt. Soc. Am.* A **13** 967
[4] van Enk S J and Nienhuis G 1992 *Opt. Commun.* **94** 147
[5] Nienhuis G and Allen L 1993 *Phys. Rev.* A **48** 656
[6] Beijersbergen M W, Allen L, van der Veen H E L O and Woerdman J P 1993 *Opt. Commun.* **96** 123132
[7] Calvo G F, Picón and Bagan E 2006 *Phys. Rev.* A **73** 13805
[8] Tasca D S, Gomes R M, Toscano F, Souto Ribeiro P H and Walborn S P 2011 *Phys. Rev.* A **83** 52325