The Physics of Exceptional Points

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Abstract. A short resume is given about the nature of EPs followed by a discussion about their ubiquitous occurrence in a great variety of physical problems. EPs feature in quantum phase transition, quantum chaos, they produce dramatic effects in multichannel scattering, specific time dependence and more. In nuclear physics they are associated with instabilities and affect approximation schemes. EPs could be of interest for weakly bound states such as halos and nuclei along the drip line.

1. Introduction

Exceptional points (EPs) are spectral singularities that occur generically in eigenvalue problems depending on a parameter [1]. This implies classical as well as quantum mechanical cases. In the simplest case they are studied in two-dimensional matrix problems [2, 3]. They are of physical interest as there is a great variety of physical situations where the singularities explain particular, in some cases dramatic effects [1]. Below we briefly present the formal background followed by a description of the first physical manifestation of the mathematical properties. The subsequent sections are devoted to some of the major physical cases where EPs play a direct role in the understanding of specific phenomena.

2. Exceptional Points

For a two-dimensional matrix the phenomenon of level repulsion is easily demonstrated. Consider the problem

$$\begin{aligned} H(\lambda) &= H_0 + H_1(\lambda) = H_0 + \lambda V \\ &= \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} + \lambda \begin{pmatrix} \epsilon_1 & \delta \\ \delta & \epsilon_2 \end{pmatrix} \end{aligned} \tag{1}$$

where the parameters ω_k and ϵ_k determine the non-interacting resonance energies $E_k = \omega_k + \lambda\epsilon_k$, $k = 1, 2$. Owing to the interaction invoked by the matrix element δ the two levels do not cross but repel each other. In fact, the levels turn out to be

$$E_{1,2}(\lambda) = \frac{1}{2}(\omega_1 + \omega_2 + \lambda(\epsilon_1 + \epsilon_1) \mp D \tag{2}$$

$$D = \sqrt{CC(\lambda - EP1)(\lambda - EP2))} \tag{3}$$

$$CC = 4\delta^2 + (\epsilon_1 - \epsilon_2)^2 \tag{4}$$

[1] see also the workshop at Stellenbosch in November 2010: **http://www.nithep.ac.za/2g6.htm**

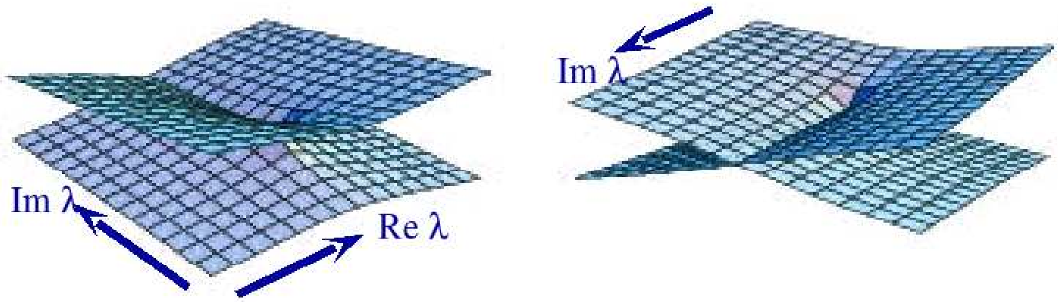

Figure 1. Perspective view of the Riemann sheet structure of two coalescing energy levels in the complex $\lambda-$plane

and the two levels *coalesce* for complex values of λ in the vicinity of the level repulsion, that is at

$$EP1 = \frac{i(\omega_1 - \omega_2)}{-2\delta - i(\epsilon_1 - \epsilon_2)} \tag{5}$$

$$EP2 = \frac{i(\omega_1 - \omega_2)}{+2\delta - i(\epsilon_1 - \epsilon_2)}. \tag{6}$$

We use the term *coalesce* as the pattern is distinctly different from a usual degeneracy encountered for hermitian operators. Note that $H(\lambda)$ is not hermitian for complex values of λ, it thus requires an open system to approach an EP in the laboratory. The difference between the hermitian and the non-hermitian case is clearly manifested by the occurrence of only *one* eigenvector (instead of the two in the case of a genuine degeneracy). The only one eigenvector is here given by

$$|\phi_{EP1}\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \tag{7}$$

$$|\phi_{EP2}\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{8}$$

independent of parameters. Note that the norm - that is the scalar product $\langle\tilde{\phi}_{EPk}|\phi_{EPk}\rangle, k = 1, 2$ - vanishes. It is the square root type of behaviour of the eigenvalues - implying an infinite derivative in the variable λ - and the vanishing norm of the likewise coalescing eigenfunctions that invoke specific observable effects.

3. Observable effects

Many cases of specific effects have been reported in the literature during the past ten years. We here can discuss only a few in some detail.

3.1. Microwave cavity

Probably the first time ever the direct encircling of the square root branch point - that is the manifestation of the two Riemann sheets (see Fig.1) - was accomplished with a microwave resonator [4].

The realisation of the complex parameter λ was implemented in the laboratory by two real parameters: (i) the coupling between the two halves of the cavity and (ii) the variation of the one level in one half of the cavity. In the experiment the direct approach of the EP was avoided while the encircling was done at close distance. One encirclement clearly swapped the levels and

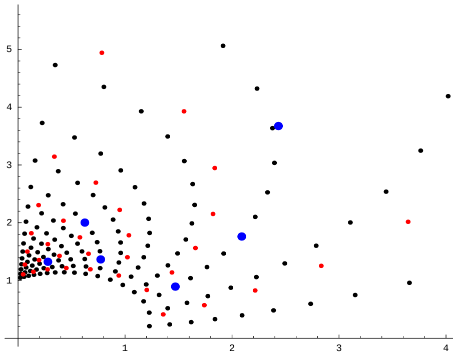

Figure 2. Exceptional points in the λ-plane for $N = 8$ (blue), $N = 16$ (red) and $N = 32$ (black).

so did the corresponding wave functions that were measured as well. Moreover, one of the wave function picks up a phase, i.e. after one round one obtains $|\phi_1\rangle \to -|\phi_2\rangle$ and $|\phi_2\rangle \to |\phi_1\rangle$. As a consequence, it needs four rounds for the wave functions to recover the original configuration, in other words one obtains the pattern for subsequent encircling

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}-\phi_2\\ \phi_1\end{pmatrix} \to \begin{pmatrix}-\phi_1\\ -\phi_2\end{pmatrix} \to \begin{pmatrix}\phi_2\\ -\phi_1\end{pmatrix} \to \begin{pmatrix}\phi_1\\ \phi_1\end{pmatrix}.$$

This sequence has been predicted and was established experimentally. It confirms a forth order root for the normalised wave functions (recall: the norm vanishes at the EP, the leading order is $\sim \sqrt{\lambda - \lambda_{EP}}$). Note that the sequence has a chiral property: going clockwise yields a result different from the one going counterclockwise.

This chiral property of the wave function at the EP has been confirmed in a second experiment [5] where the phase difference of $\pi/2$ between the two components (see (7) or (8)) has been confirmed in a direct approach of an EP. For further details see [4, 5].

The same results have been reconfirmed with two coupled electronic circuits [6].

3.2. Quantum phase transitions, chaos

The Lipkin model [7] is a toy model often used to study quantum phase transitions. The interaction of the two level model lifts or lowers a Fermion pair between the two levels. For N particles it can be formulated in terms of the angular momentum operators and reads

$$H(\lambda) = J_z + \frac{\lambda}{N}(J_+^2 + J_-^2) \tag{9}$$

with $J_z, J_\pm$ being the N-dimensional representations of the $SU(2)$ operators. There is a phase transition at $\lambda > 1$ that moves toward $\lambda = 1$ in the thermodynamic limit. The Hamiltonian has an inherent symmetry: even and odd numbers k of the levels E_k do not interact. The phase for $\lambda < 1$ is the 'normal' phase where the symmetry of the problem is preserved by the levels and wave functions. In the 'deformed' phase for $\lambda > 1$ the symmetry is broken in that even and odd k become degenerate. Here the role of the EPs is crucial to bring about the phase change in the spectrum [8]. In Fig.2 the pattern of the EPs is illustrated for low values of N. It is clearly seen how the EPs accumulate for increasing N on the real axis with the tendency to move towards the point $\lambda = 1$. The spectrum remains unaffected by singularities in the region of the normal phase while it is strongly affected around the critical point. For finite temperature these singularities feature accordingly in the partition function [9]. If the model is perturbed the regular pattern of the EPs is destroyed and so is accordingly the spectrum. The onset of chaos [10] is clearly discernible in the region of the phase transition while the model remains robust outside the critical region for sufficiently mild perturbation.

3.3. *The role of EPs in approximation schemes*

The well known Random Phase Approximation (RPA) used in many body problems yields an effective Hamiltonian that is non-hermitian [11]. As a result, eigenvalues are not necessarily real. Depending on the strength of the, say, particle-hole interaction two real eigenvalues Ω and $-\Omega$ coalesce at $\Omega = 0$ and move then into the complex plane when the interaction is increased. Often this instability point is associated with a phase transition of the underlying mean field [11, 12]. It is an EP with all its characteristics: square root branch point in the interaction strength and the vanishing norm of the wave function.

A perturbative approach in shell model calculations can be hampered by singularities associated with *intruder states* [13]. These singularities are EPs where two levels coalesce thus limiting the radius of convergence of the perturbation series.

Recent approaches to model nuclei on the drip line [14] use resonance states to describe the continuum. The coalescence of two resonances can invoke specific physical effects owing to the strong increase of the associated spectroscopic factors being caused by the vanishing norm of the wave functions at the EP.

3.4. *The symmetry breaking point for $\mathcal{PT}$-symmetric Hamiltonians*

It has been suggested to extent the class of the traditional hermitian Hamiltonians by a specific choice of non-hermitian operators [15]. Hamiltonians that are symmetric under the combined operation of parity and time reversal transformation, the $\mathcal{PT}$-symmetric operators, can have a real spectrum even though the operators can be non-hermitian. It turns out that if the eigenstates preserve the symmetry, the eigenvalues are real, while for symmetry breaking the eigenvalues are complex [16]. The points where this symmetry is broken are the EPs of the problem. In the meanwhile, while there is plenty theoretical literature on this subject [17, 18], there is beautiful experimental evidence with optical cavities [19], optical lattices [20] and propagation of light [21].

3.5. *EPs and Feshbach resonance in atomic/molecular physics*

Using Feshbach resonance techniques there are recent proposals for resonant dissociation by lasers of H_2^+ molecules or alkali dimers where the effects of EPs are expected to feature prominently [22]. Similar in spirit, a Bose-Einstein condensate of neutral atoms with induced electromagnetic attractive $(1/r)$ interaction has been discussed recently as another system allowing a tunable interaction [23]. The critical value - an EP - where the onset of the collapse of the condensate occurs is interpreted as a transition point from separate atoms to the formation of molecules or clusters [24].

3.6. *Special effects in multichannel scattering*

Depending on a judicious choice of parameters the proximity of EPs can invoke dramatic effects in multichannel scattering such as a sudden increase of the cross section in one channel, even by orders of magnitude. In turn, a second channel is suppressed and can show a resonance curve that deviates substantially from the usual Lorentz shape [25]. Related to this behaviour is the pattern in the time domain [26]. Depending on the initial conditions the wave function displays characteristic features such as very fast decay or the opposite, i.e. very long life time. At the EP the wave function typically has a linear term in time besides the usual exponential behaviour.

4. Summary

The ubiquitous occurrence of EPs in all eigenvalue problems that depend on a parameter can have significant and often dramatic effects of observables in a great variety of physical phenomena. A few decades ago, these singularities appeared as a purely mathematical feature that could cause problems in approximation schemes. I was only about ten years ago that their physical manifestation has been demonstrated in experiments that were basically classical in nature (recall that an EP can be approached in the laboratory only in an open system). More recently, there are now definite theoretical and experimental proposals in atomic and molecular physics, using lasers for triggering and measuring specific transitions. In nuclear physics, where there is now great interest in open systems, that is in nuclei on the drip line, the coalescence of resonance states is expected to produce specific effects such as enhancements of particular reactions.

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