

Alternative derivation of the master equation for a particle in an external field subject to continuous measurement

K Garapo¹ and T Konrad¹

¹ School of Physics, University of KwaZulu-Natal, Westville, Durban 4000, South Africa.

E-mail: 209523532@ukzn.ac.za and konradt@ukzn.ac.za

Abstract. The theory of continuous measurement provides a tool to monitor the evolution of the wave function of a single quantum system in real time. We re-derive the master equation in the non-selective regime for the dynamics of the wave function of a particle in an external potential which is subject to continuous measurement of position. In the derivation we view continuous measurement as the limit of a sequence of unsharp position measurements. Unsharp position measurements are achieved by selecting generalised measurement observables, or in mathematical terms, positive operator valued measures (POVM) rather than the standard von Neumann projection operators which are a special class of the sub-class of POVM's called projection valued measures (PVM). We also introduce a commutative algebra that allows us to perform commutative operations with non-commuting position operators. We then deduce the stochastic Ito equations for the selective regime of measurement.

1. Introduction

In 1987, Caves and Milburn [1] suggested a model for the continuous measurement of the position of a quantum system. Their model was based on the theory of continuous quantum measurement as suggested, in 1982, by Barchielli [2] et al. In 1988, Diosi [3] showed that continuous measurement of position in the selective regime can be represented by a certain Ito stochastic master equation. In this paper we take a simplified approach to re-derive the master equations for continuous position measurement in both the selective and non-selective regimes. In our approach, we view continuous position measurement as a sequence of unsharp measurements of position of a quantum system in the following limit;

$$\lim_{\substack{\tau \rightarrow 0 \\ \sigma \rightarrow \infty}} \frac{1}{\sigma^2 \tau} = \gamma, \quad (1)$$

where τ is the time interval between two consecutive measurements, σ is the precision parameter of the measurements and γ is a finite quantity called the decoherence rate. For historical reasons we shall refer to this limit as the Barchielli limit. We represent the unsharp measurement of position of a quantum system by generalised position observables.

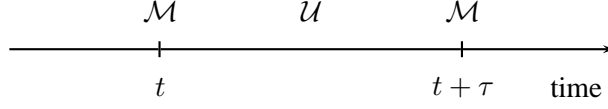


Figure 1. Schematic diagram for the time evolution of a system undergoing a sequence of measurements \hat{M} at time intervals τ . Between two consecutive measurements the closed system evolves unitarily.

2. Non-selective regime

We consider weak position measurements of a system with one spatial degree of freedom x , with outcomes \bar{x} , represented by Kraus operators [4]

$$\hat{M}_{\bar{x}} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\hat{x} - \bar{x})^2}{4\sigma^2}\right), \quad (2)$$

and unitaries

$$\hat{U} = \exp\left(-\frac{i}{\hbar}\hat{H}\tau\right) = \mathbb{1} - \frac{i}{\hbar}\hat{H}\tau + \mathcal{O}(\tau^2), \quad (3)$$

where \hat{H} is the Hamiltonian of the system, \hat{x} is the position measurement operator and $\hbar = h/2\pi$ (Planck's constant divided by 2π). The effects $\hat{E}_{\bar{x}} = \hat{M}_{\bar{x}}^\dagger \hat{M}_{\bar{x}}$ of the measurements are Gaussian. If at time t , the state of the system is represented by the density operator $\hat{\rho}(t)$, then after a time τ , the state of the system is given by (compare figure 1.)

$$\begin{aligned} \hat{\rho}(t + \tau) &= \int_{-\infty}^{\infty} d\bar{x} \hat{U} \hat{M}_{\bar{x}} \hat{\rho}(t) \hat{M}_{\bar{x}}^\dagger \hat{U}^\dagger \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \hat{U} \left(\int_{-\infty}^{\infty} d\bar{x} \exp\left(-\frac{(\hat{x} - \bar{x})^2}{4\sigma^2}\right) \hat{\rho}(t) \exp\left(-\frac{(\hat{x} - \bar{x})^2}{4\sigma^2}\right) \right) \hat{U}^\dagger. \end{aligned} \quad (4)$$

In order to carry out the integration we introduce a commutative super-algebra [5, 6] with position operators \hat{x}_L and \hat{x}_R which are defined as follows;

$$\hat{x}_L \hat{\rho}(t) \equiv \hat{\rho}(t) \hat{x}_L := \hat{x} \hat{\rho}(t) \quad (5)$$

and

$$\hat{x}_R \hat{\rho}(t) \equiv \hat{\rho}(t) \hat{x}_R := \hat{\rho}(t) \hat{x}. \quad (6)$$

Given that the operator \hat{x} has the following spectral representation,

$$\hat{x} = \int_{-\infty}^{\infty} x \hat{P}(dx) \quad (7)$$

where x are position eigenvalues and $\hat{P}(dx) = dx |x\rangle\langle x|$ is a projection valued measure. In a similar way we can expand the operators \hat{x}_L and \hat{x}_R as follows;

$$\hat{x}_L = \int_{-\infty}^{\infty} x' \hat{P}_L(dx') \quad \text{and} \quad \hat{x}_R = \int_{-\infty}^{\infty} x'' \hat{P}_R(dx''). \quad (8)$$

The actions of $\hat{P}_L(dx')$ and $\hat{P}_R(dx'')$ on any arbitrary operator \hat{A} on the Hilbert space \mathcal{H} are defined as follows;

$$\hat{P}_L(dx')\hat{A} \equiv \hat{A}\hat{P}_L(dx') := \hat{P}(dx)\hat{A}, \quad (9)$$

and

$$\hat{P}_R(dx'')\hat{A} \equiv \hat{A}\hat{P}_R(dx'') := \hat{A}\hat{P}(dx) \quad (10)$$

respectively. The equations (9) and (10) are consistent with equations (5) and (6). Since the operators \hat{x}_L and \hat{x}_R commute with all operators on the Hilbert space we can rewrite equation (4) as follows;

$$\begin{aligned} \hat{\rho}(t + \tau) &= \frac{1}{\sqrt{2\pi\sigma^2}} \hat{U} \left(\int_{-\infty}^{\infty} d\bar{x} \exp \left(\frac{-(\hat{x}_L - \bar{x})^2 - (\hat{x}_R - \bar{x})^2}{4\sigma^2} \right) \right) \hat{\rho}(t) \hat{U}^\dagger \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \hat{U} \left(\int_{-\infty}^{\infty} d\bar{x} \exp \left(-\frac{1}{2\sigma^2} \bar{x}^2 + \frac{\hat{x}_L + \hat{x}_R}{2\sigma^2} \bar{x} - \frac{\hat{x}_L^2 + \hat{x}_R^2}{4\sigma^2} \right) \right) \hat{\rho}(t) \hat{U}^\dagger. \end{aligned} \quad (11)$$

We shall evaluate the integral in equation (11) as follows;

$$\begin{aligned} &\int_{-\infty}^{\infty} d\bar{x} \exp \left(-\frac{1}{2\sigma^2} \bar{x}^2 + \frac{\hat{x}_L + \hat{x}_R}{2\sigma^2} \bar{x} - \frac{\hat{x}_L^2 + \hat{x}_R^2}{4\sigma^2} \right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} d\bar{x} \exp \left(-\frac{1}{2\sigma^2} \bar{x}^2 + \frac{x' + x''}{2\sigma^2} \bar{x} - \frac{(x')^2 + (x'')^2}{4\sigma^2} \right) \right] \hat{P}_L(dx') \hat{P}_R(dx'') \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sqrt{2\pi\sigma^2} \left(1 - \frac{1}{8\sigma^2} ((x')^2 + (x'')^2 - 2x'x'') + \mathcal{O}(\sigma^{-4}) \right) \right] \hat{P}_L(dx') \hat{P}_R(dx'') \\ &= \sqrt{2\pi\sigma^2} \left(\mathbb{1} - \frac{1}{8\sigma^2} (\hat{x}_L^2 + \hat{x}_R^2 - 2\hat{x}_L\hat{x}_R) + \mathcal{O}(\sigma^{-4}) \right), \end{aligned} \quad (12)$$

where the integration over the possible measurement results \bar{x} has been carried out and the obtained exponential Taylor-expanded. Substituting equations (3) and (12) into equation (11) yields the following;

$$\begin{aligned} \Delta\hat{\rho}(t) &= \hat{\rho}(t)(t + \tau) - \hat{\rho}(t) = \left(\mathbb{1} - \frac{i}{\hbar} \hat{H}\tau + \mathcal{O}(\tau^2) \right) \\ &\quad \times \left(\mathbb{1} - \frac{1}{8\sigma^2} (\hat{x}_L^2 + \hat{x}_R^2 - 2\hat{x}_L\hat{x}_R) + \mathcal{O}(\sigma^{-4}) \right) \hat{\rho}(t) \left(\mathbb{1} + \frac{i}{\hbar} \hat{H}\tau + \mathcal{O}(\tau^2) \right) - \hat{\rho}(t) \\ &= -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)]\tau - \frac{1}{8\sigma^2} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{i\tau}{8\hbar\sigma^2} [\hat{H}, [\hat{x}, [\hat{x}, \hat{\rho}(t)]]] + \mathcal{O}(\tau^2) + \mathcal{O}(\sigma^{-4}). \end{aligned} \quad (13)$$

In the Barchielli limit, equation (13) becomes

$$d\hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)]dt - \frac{\gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]]dt. \quad (14)$$

We note that the higher order terms in τ vanish as τ approaches 0. Equation (14) is the master equation of continuous position measurement in the non-selective regime.

3. Selective Regime

While we average the state of the system after measurement over all possible outcomes in the non-selective regime, we have to account for the measurement results in the selective regime. If at a time t the state of the system is represented by the density operator $\hat{\rho}(t)$, then after a time τ the state of the system is given by

$$\hat{\rho}(t + \tau) = \frac{1}{p_{\hat{\rho}}(\bar{x})} \hat{U} \hat{M}_{\bar{x}} \hat{\rho}(t) \hat{M}_{\bar{x}}^{\dagger} \hat{U}^{\dagger}, \quad (15)$$

where $p_{\hat{\rho}}(\bar{x})$ is the probability of obtaining the measurement result \bar{x} given that the state of the system is $\hat{\rho}$. The inverse of the probability is evaluated as follows;

$$\begin{aligned} \frac{1}{p_{\hat{\rho}}(\bar{x})} &= \left[\text{tr} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{(\hat{x} - \bar{x})^2}{2\sigma^2} \right) \hat{\rho}(t) \right\} \right]^{-1} \\ &\approx \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{\bar{x}^2}{2\sigma^2} \right) \left(1 + \frac{\bar{x}}{\sigma^2} \langle \hat{x} \rangle_{\hat{\rho}} \right) \right]^{-1} \\ &\approx \sqrt{2\pi\sigma^2} \exp \left(\frac{\bar{x}^2}{2\sigma^2} \right) \left(1 - \frac{\bar{x}}{\sigma^2} \langle \hat{x} \rangle_{\hat{\rho}} + \frac{\bar{x}^2}{\sigma^4} \langle \hat{x} \rangle_{\hat{\rho}}^2 \right), \end{aligned} \quad (16)$$

and the measurement operator is expanded as follows;

$$\hat{M}_{\bar{x}} = \frac{1}{\sqrt{4\pi\sigma^2}} \exp \left(-\frac{\bar{x}^2}{4\sigma^2} \right) \left(\mathbb{1} - \frac{\hat{x}^3 \bar{x}}{8\sigma^4} + \frac{\hat{x}^2 \bar{x}^2}{8\sigma^4} + \frac{\hat{x} \bar{x}}{2\sigma^2} + \frac{\hat{x}^4}{32\sigma^4} - \frac{\hat{x}^2}{4\sigma^2} + \mathcal{O}(\sigma^{-6}) \right). \quad (17)$$

In doing both expansions we take note of the fact that except for terms in \bar{x}^2/σ^{-4} all terms in σ^{-4} and below vanish in the Barchielli limit. To simplify the evaluation of equation (15), we first evaluate the measurement part.

$$\begin{aligned} &\frac{\hat{M}_{\bar{x}} \hat{\rho}(t) \hat{M}_{\bar{x}}^{\dagger}}{p_{\hat{\rho}}(\bar{x})} \\ &= \left(1 - \frac{\bar{x}}{\sigma^2} \langle \hat{x} \rangle_{\hat{\rho}} + \frac{\bar{x}^2}{\sigma^4} \langle \hat{x} \rangle_{\hat{\rho}}^2 \right) \left(\mathbb{1} - \frac{\hat{x}^3 \bar{x}}{8\sigma^4} + \frac{\hat{x}^2 \bar{x}^2}{8\sigma^4} + \frac{\hat{x} \bar{x}}{2\sigma^2} + \frac{\hat{x}^4}{32\sigma^4} - \frac{\hat{x}^2}{4\sigma^2} + \mathcal{O}(\sigma^{-6}) \right) \hat{\rho}(t) \\ &\quad \times \left(\mathbb{1} - \frac{\hat{x}^3 \bar{x}}{8\sigma^4} + \frac{\hat{x}^2 \bar{x}^2}{8\sigma^4} + \frac{\hat{x} \bar{x}}{2\sigma^2} + \frac{\hat{x}^4}{32\sigma^4} - \frac{\hat{x}^2}{4\sigma^2} + \mathcal{O}(\sigma^{-6}) \right) \\ &= \left(1 - \frac{\bar{x}}{\sigma^2} \langle \hat{x} \rangle_{\hat{\rho}} + \frac{\bar{x}^2}{\sigma^4} \langle \hat{x} \rangle_{\hat{\rho}}^2 \right) \left(\hat{\rho}(t) + \frac{\bar{x}}{2\sigma^2} \{ \hat{x}, \hat{\rho}(t) \} - \frac{1}{8\sigma^2} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] - \frac{1}{8\sigma^2} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} \right. \\ &\quad + \frac{\bar{x}^2}{8\sigma^4} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} - \frac{\bar{x}}{8\sigma^4} \{ \hat{x}^3, \hat{\rho}(t) \} + \frac{1}{32\sigma^4} \{ \hat{x}^4, \hat{\rho}(t) \} \\ &\quad \left. - \frac{\bar{x}}{4\sigma^4} \hat{x}^2 \hat{\rho}(t) \hat{x} + \frac{1}{16\sigma^4} \hat{x}^2 \hat{\rho}(t) \hat{x}^2 + \mathcal{O}(\sigma^{-6}) \right) \\ &= \left(1 - \bar{x} \tau \gamma \langle \hat{x} \rangle_{\hat{\rho}} + \bar{x}^2 \tau^2 \gamma^2 \langle \hat{x} \rangle_{\hat{\rho}}^2 \right) \left(\hat{\rho}(t) + \frac{\bar{x} \tau \gamma}{2} \{ \hat{x}, \hat{\rho}(t) \} - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] \right. \\ &\quad - \frac{\tau \gamma}{8} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} + \frac{\bar{x}^2 \tau^2 \gamma^2}{8} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} - \frac{\bar{x} \tau^2 \gamma^2}{8} \{ \hat{x}^3, \hat{\rho}(t) \} + \frac{\tau^2 \gamma^2}{32} \{ \hat{x}^4, \hat{\rho}(t) \} \\ &\quad \left. - \frac{\bar{x} \tau^2 \gamma^2}{4} \hat{x}^2 \hat{\rho}(t) \hat{x} + \frac{\tau^2 \gamma^2}{16} \hat{x}^2 \hat{\rho}(t) \hat{x}^2 + \mathcal{O}(\tau^3) \right), \end{aligned} \quad (18)$$

in the Barchielli limit. We now take the approach of [3] and [7] and introduce stochastic equations that govern our measurement results \bar{x} . From [3] we know that $\bar{x}_t = \langle \hat{x} \rangle_{\hat{\rho}(t)} + \frac{1}{\sqrt{\gamma}} w_t$, where w_t is standard white noise which is defined by $\langle w_t \rangle_{st} = 0$ and $\langle w_t w_s \rangle_{st} = \delta(t - s)$. [3] mentions the mathematical

shortcomings of the equation and introduces the mathematically well behaved quantity Q_t which is the time integrated measurement signal, given by

$$Q_t = \int_0^t \bar{x}_{t'} dt' = \int_0^t \left(\langle \hat{x} \rangle_{\hat{\rho}(t')} + \frac{1}{\sqrt{\gamma}} w_{t'} \right) dt'. \quad (19)$$

The integral in equation (19) is generally not easy to evaluate and hence we prefer the differential form,

$$dQ_t = \langle \hat{x} \rangle_{\hat{\rho}(t)} dt + \gamma^{-\frac{1}{2}} dW_t, \quad (20)$$

where $W_t = \int_0^t w_{t'} dt'$ is a Wiener process. The Wiener increments dW_t satisfy the following Ito rules; $\langle dW_t \rangle_{st} = 0$, $(dW_t)^2 = dt$, and $(dW_t)^n = 0$ for $n > 2$. In the Barchielli limit, $\bar{x}\tau = dQ_t$. It follows that $\bar{x}^2\tau^2 = (dQ_t)^2 = \frac{1}{\gamma} dt$, after applying Ito rules. Equation (18) thus reduces to

$$\begin{aligned} & \frac{\hat{M}_{\bar{x}} \hat{\rho}(t) \hat{M}_{\bar{x}}^\dagger}{p_{\hat{\rho}}(\bar{x})} \\ &= \left(1 - (dQ_t) \gamma \langle \hat{x} \rangle_{\hat{\rho}} + \tau \gamma \langle \hat{x} \rangle_{\hat{\rho}}^2 \right) \left(\hat{\rho}(t) + \frac{(dQ_t) \gamma}{2} \{ \hat{x}, \hat{\rho}(t) \} - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] \right) \\ & \quad - \frac{\tau \gamma}{8} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} + \frac{\tau \gamma^2}{8 \gamma} \{ \hat{x}, \{ \hat{x}, \hat{\rho}(t) \} \} \\ &= \left(1 - (dQ_t) \gamma \langle \hat{x} \rangle_{\hat{\rho}} + \tau \gamma \langle \hat{x} \rangle_{\hat{\rho}}^2 \right) \left(\hat{\rho}(t) + \frac{(dQ_t) \gamma}{2} \{ \hat{x}, \hat{\rho}(t) \} - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] \right) \\ &= \hat{\rho}(t) - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{(dQ_t) \gamma}{2} \{ \hat{x}, \hat{\rho}(t) \} - (dQ_t) \gamma \hat{\rho}(t) \langle \hat{x} \rangle_{\hat{\rho}} \\ & \quad - \frac{\tau \gamma \langle \hat{x} \rangle_{\hat{\rho}}}{2} \{ \hat{x}, \hat{\rho}(t) \} + \tau \gamma \langle \hat{x} \rangle_{\hat{\rho}}^2 \hat{\rho}(t) \\ &= \hat{\rho}(t) - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{(dQ_t) \gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} - \frac{\tau \gamma \langle \hat{x} \rangle_{\hat{\rho}}}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} \\ &= \hat{\rho}(t) - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{\gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} (dQ_t - \tau \langle \hat{x} \rangle_{\hat{\rho}}). \end{aligned} \quad (21)$$

From equation (21) it follows that

$$\begin{aligned} & \hat{\rho}(t + \tau) \\ &= \left(\mathbf{1} - \frac{i}{\hbar} \hat{H} \tau + \mathcal{O}(\tau^2) \right) \left(\hat{\rho}(t) - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{\gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} (dQ_t - \tau \langle \hat{x} \rangle_{\hat{\rho}}) \right) \\ & \quad \times \left(\mathbf{1} + \frac{i}{\hbar} \hat{H} \tau + \mathcal{O}(\tau^2) \right) \\ &= \hat{\rho}(t) - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{\gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} (dQ_t - \tau \langle \hat{x} \rangle_{\hat{\rho}}) - \frac{i}{\hbar} \hat{H} \tau \hat{\rho}(t) + \hat{\rho}(t) \frac{i}{\hbar} \hat{H} \tau \\ &= \hat{\rho}(t) - \frac{i \tau}{\hbar} [\hat{H}, \hat{\rho}(t)] - \frac{\tau \gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] + \frac{\gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} (dQ_t - \tau \langle \hat{x} \rangle_{\hat{\rho}}), \end{aligned} \quad (22)$$

and

$$d\hat{\rho}(t) = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)] dt - \frac{\gamma}{8} [\hat{x}, [\hat{x}, \hat{\rho}(t)]] dt + \frac{\gamma}{2} \{ \hat{x} - \langle \hat{x} \rangle_{\hat{\rho}}, \hat{\rho}(t) \} (dQ_t - \langle \hat{x} \rangle_{\hat{\rho}} dt). \quad (23)$$

Equation (23) is the master equation for continuous measurement of position in the selective regime. We refer readers who wish to have in-depth understanding of the stochastic calculus we used in this derivation to the book by Gardiner [8].

4. Conclusion

We re-derived the master equations for the continuous measurement of position in both the selective and non-selective regimes. In the derivation of the master equations we applied a commutative super-algebra and the Ito stochastic calculus, which was suggested by Diosi [3]. In contrast to Diosi, our approach is based on combining the Kraus representation of the state change due to measurement with the Ito calculus by expressing the integrated measurement signal Q by means of a Wiener process. This leads to the simplification of the derivation. The derived master equations are important tools to describe state monitoring and control of individual quantum systems.

Acknowledgement

The authors would like to thank Dr. S. K. Choudhary for helpful discussions and remarks.

References

- [1] Caves C M and Milburn G J 1987 *Phys. Rev. A* **36** 5543
- [2] Barchielli A, Lanz L and Prosperi G 1982 *Il Nuovo Cimento B (1971-1996)* **72**(1) 79
- [3] Diosi L 1988 *Phys. Lett. A* **129** 419
- [4] Kraus K 1983 *Lecture Notes in Physics* **190**
- [5] Audretsch J, Diósi L and Konrad T 2003 *Phys. Rev. A* **68** 034302
- [6] Konrad T 2003 Ph.D. thesis Universitat Konstanz
- [7] Diosi L, Konrad T, Scherer A and Audretsch J 2006 (*Preprint quant-ph/0608097*)
- [8] Gardiner C W 1985 *Handbook of stochastic methods for physics, chemistry, and the natural sciences* vol 2nd (Springer-Verlag)