Designing Reservoir for $1/t$ decoherence process in Jaynes-Cummings model

Filippo Giraldi

E-mail: giraldi@ukzn.ac.za,filgi@libero.it
Quantum Research Group, School of Physics and National Institute for Theoretical Physics, University of KwaZulu-Natal, Durban 4001, South Africa

Francesco Petruccione

E-mail: petruccione@ukzn.ac.za
Quantum Research Group, School of Physics and National Institute for Theoretical Physics, University of KwaZulu-Natal, Durban 4001, South Africa

Abstract. Decoherence indicates the process that a quantum system undergoes through the interaction with its external environment. In a typical situation, a two-level system (qubit) interacting with a Lorentzian type continuous distributions of field modes, one observes exponential relaxation of the reduced density matrix. In this Jaynes-Cummings model scenario, a special class of reservoirs is designed in order to control or delay the destructive effect of the environment on qubit coherence. In this way, decoherence processes slower than the usual exponential ones are obtained: over estimated long time scales, we demonstrate inverse power law relaxations with powers decreasing continuously to unity according to the choice of the particular reservoir. The designed reservoirs exhibit a photonic band gap coinciding with the qubit transition frequency and are piecewise similar to those usually adopted in Quantum Optics, i.e., sub-ohmic at low frequencies and inverse power laws at high frequencies. Initially, the reservoir is assumed to be in the vacuum state and is unentangled from the qubit versing in a generic state. The exact dynamics results to be described by series of Fox H-functions. The simple form of the designed reservoir can be accessible experimentally.

1. Introduction

Decoherence indicates the process that a quantum system undergoes through the interaction with its external environment. Great attention has been devoted to the dynamics of a qubit, or, alternatively, a two-level system (TLS), coupled to its external environment [1, 2], given the many applications in quantum optics, quantum information, atom-cavity interactions, molecular dynamics, spectroscopy, and solid state physics. The environment is represented by a reservoir of bosons [3], characterized by the corresponding spectral density function. The dependance of decoherence on the explicit form of the spectral density function has been studied for sub-ohmic, ohmic, super-ohmic and Lorentzian forms of the spectral density [4, 3], by adopting the Jaynes-Cumming model [5].

In the following we consider the Jaynes-Cummings model and study a special class of reservoirs piecewise similar to those usually adopted, e.g., sub-ohmic and Lorentzian ones. We will show, that the specially designed reservoirs strongly suppress the decoherence. In particular, we mention that the following analytical calculations concerning the exact dynamics of the model are performed for a positive range of modes frequencies.

By choosing $\hbar = 1$, the Hamiltonian of the total system is $H_S + H_E + H_I$, where the Hamiltonian of the system, H_S, the Hamiltonian of the environment H_E and the interaction Hamiltonian H_I are given by

$$H_S = \omega_0\, \sigma_+ \sigma_-, \qquad H_E = \sum_{k=1}^{\infty} \omega_k\, a_k^{\dagger} a_k, \qquad H_I = \sum_{k=1}^{\infty} \left(g_k\, \sigma_+ \otimes a_k + g_k^{*}\, \sigma_- \otimes a_k^{\dagger} \right).$$

The rising and lowering operators, σ_+ and σ_-, respectively, act on the Hilbert space of the qubit, defined through the equalities $\sigma_+ = \sigma_-^{\dagger} = |1\rangle\langle 0|$, while $a_k^{\dagger}$ and a_k are the creation and annihilation operators, respectively, acting on the Hilbert space of the k-th boson, fulfilling the commutation rule $\left[a_k, a_k'^{\dagger}\right] = \delta_{k,k'}$ for every $k, k' = 1, 2, 3, \ldots$. The parameters g_k represent the coupling between the transition $|0\rangle \leftrightarrow |1\rangle$ and the k-th mode of the radiation field, while ω_0 is the qubit transition frequency.

The whole system is initially set in the following ket state:

$$|\Psi(0)\rangle = \left(c_0|0\rangle + c_1(0)|1\rangle\right) \otimes |0\rangle_E, \tag{1}$$

where $|0\rangle_E$ is the vacuum state of the environment. The exact time evolution is described by the form

$$|\Psi(t)\rangle = c_0|0\rangle \otimes |0\rangle_E + c_1(t)|1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} b_k(t)|0\rangle \otimes |k\rangle_E, \;\; |k\rangle_E = a_k^{\dagger}|0\rangle_E, \qquad k = 0, 1, 2, \ldots.$$

The dynamics is easily studied in the interaction picture,

$$|\Psi(t)\rangle_I = e^{\imath (H_S + H_E)t}|\Psi(t)\rangle = c_0|0\rangle \otimes |0\rangle_E + C_1(t)|1\rangle \otimes |0\rangle_E + \sum_{k=1}^{\infty} B_k(t)|0\rangle \otimes |k\rangle_E,$$

where $\imath$ is the imaginary unity, $C_1(t) = e^{\imath\omega_0 t}\, c_1(t)$ and $B_k(t) = e^{\imath\omega_k t}\, b_k(t)$ for every $k = 1, 2, \ldots$. The Schrödinger equation gives the forms:

$$\dot{C}_1(t) = -\imath \sum_{k=1}^{\infty} g_k\, B_k(t)\, e^{-\imath(\omega_k - \omega_0)t}, \qquad \dot{B}_k(t) = -\imath\, g_k^{*}\, C_1(t)\, e^{\imath(\omega_k - \omega_0)t},$$

leading to the following convoluted structure equation for the amplitude $\langle 1| \otimes {}_E\langle 0||\Psi(t)\rangle_I$, labeled as $C_1(t)$,

$$\dot{C}_1(t) = -\left(f * C_1\right)(t), \tag{2}$$

where f is the two-point correlation function of the reservoir of field modes,

$$f\left(t - t'\right) = \sum_{k=1}^{\infty} |g_k|^2\, e^{-\imath(\omega_k - \omega_0)(t - t')}.$$

For a continuous distribution of modes described by $\eta(\omega)$, the correlation function is expressed through the spectral density function $J(\omega)$,

$$f(\tau) = \int_0^{\infty} J(\omega)\, e^{-\imath(\omega - \omega_0)\tau} d\omega,$$

where $J(\omega) = \eta(\omega)\, |g(\omega)|^2$ and $g(\omega)$ is the frequency dependent coupling constant.

2. The decoherence process

The exact dynamics of the qubit is described by the time evolution of the reduced density matrix obtained by tracing over the Hilbert space of the bosons,

$$\rho_{1,1}(t) = 1 - \rho_{0,0}(t) = \rho_{1,1}(0)\ |G(t)|^2, \qquad \rho_{1,0}(t) = \rho^*_{0,1}(t) = \rho_{1,0}(0)\, e^{-\imath\omega_0 t} G(t). \tag{3}$$

The function $G(t)$, fulfilling the convolution equation

$$\dot{G}(t) = -\left(f * G\right)(t), \quad G(0) = 1, \tag{4}$$

drives the dynamics of the reduced density matrix describing the qubit: levels populations and decoherence term. At this stage, we choose to study the exact dynamics of the reduced density matrix of the qubit, interacting in rotating wave approximation with a reservoir of bosons described by the continuous spectral density

$$J_\alpha\left(\omega\right) = \frac{2A\left(\omega-\omega_0\right)^\alpha \Theta\left(\omega-\omega_0\right)}{a^2 + \left(\omega-\omega_0\right)^2}, \qquad A>0, \quad a>0, \quad 1>\alpha>0. \tag{5}$$

This simple form exhibits a photonic band gap (PBG) edge coinciding with the qubit transition frequency and has an absolute maximum M_α at the frequency Ω_α,

$$M_\alpha = J_\alpha\left(\Omega_\alpha\right) = A\,\alpha^{\alpha/2} a^{\alpha-2}\left(2-\alpha\right)^{1-\alpha/2}, \qquad \Omega_\alpha = \omega_0 + a\,\alpha^{1/2}\left(2-\alpha\right)^{1/2}.$$

The designed spectral densities are similar to those usually adopted, i.e. sub-ohmic at low frequencies, $\omega \simeq \omega_0$, and inverse power laws at high frequencies, $\omega \gg \omega_0$, similar to the Lorentzian one, though with different power,

$$J_\alpha\left(\omega\right) \sim 2A/a^2\left(\omega-\omega_0\right)^\alpha, \qquad \omega \to \omega_0^+, \qquad J_\alpha\left(\omega\right) \sim 2A\,\omega^{\alpha-2}, \qquad \omega\to+\infty.$$

The exact dynamics of the reduced density matrix, driven by the function $G(t)$, is described through the Fox H-function, defined through a Mellin-Barnes type integral in the complex domain,

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}\left(a_1,\alpha_1\right),\ldots,\left(a_p,\alpha_p\right)\\ \left(b_1,\beta_1\right),\ldots,\left(b_q,\beta_q\right)\end{matrix}\right.\right] = \frac{1}{2\pi\imath}\int_{\mathcal{C}} \frac{\Pi_{j=1}^m \Gamma\left(b_j+\beta_j s\right)\Pi_{m=1}^n\Gamma\left(1-a_l-\alpha_l s\right) z^{-s}}{\Pi_{l=n+1}^p\Gamma\left(a_l+\alpha_l s\right)\Pi_{j=m+1}^q\Gamma\left(1-b_j-\beta_j s\right)}\,ds,$$

under the conditions that the poles of the Gamma functions in the dominator, do not coincide. Also the empty products are interpreted as unity. The natural numbers m, n, p, q fulfill the constraints: $0 \leq m \leq q$, $0 \leq n \leq p$, and $\alpha_i, \beta_j \in (0,+\infty)$ for every $i = 1,\cdots,p$ and $j=1,\cdots,q$. For the sake of shortness, we refer to [6] for details on the contour path $\mathcal{C}$, the existence and the properties of the Fox H-functions. The exact dynamics corresponding to the reservoir of spectral density $J_\alpha\left(\omega\right)$, Eq. (5), is driven by $G_\alpha(t)$ reading

$$\begin{aligned} G_\alpha(t) \ &= \ \sum_{n=0}^\infty\sum_{k=0}^n \frac{(-1)^n z_\alpha^k z_0^{n-k} t^{3n-\alpha k}}{k!(n-k)!}\Bigg(H^{1,1}_{1,2}\left[z_1 t^2\left|\begin{matrix}(-n,1)\\ (0,1),(\alpha k-3n,2)\end{matrix}\right.\right] \\ &\quad - a^2t^2 H^{1,1}_{1,2}\left[z_1 t^2\left|\begin{matrix}(-n,1)\\ (0,1),(\alpha k-3n-2,2)\end{matrix}\right.\right]\Bigg). \end{aligned} \tag{6}$$

A detailed demonstration is given in Ref. [7].

Particular cases give simplified solutions. For example, the condition $A = A^{(\star)}$,

$$A^{(\star)} = \frac{a^{3-\alpha}}{\pi}\cos\left(\pi\alpha/2\right), \tag{7}$$

corresponding to $z_1 = 0$, gives a power series solution,

$$G_{\alpha}^{(\star)}(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \, n! \, z_{\alpha}^k \, z_0^{n-k} \, t^{3n-\alpha k}}{k! \, (n-k)! \, \Gamma(3n - \alpha k + 1)} \left\{ 1 - a^2 \, \frac{\Gamma(3n - \alpha k + 1)}{\Gamma(3n - \alpha k + 3)} \, t^2 \right\}, \quad 1 > \alpha > 0. \tag{8}$$

If the parameter α takes rational values, p/q, where p and q are distinct prime numbers such that $0 < p < q$, the solution of Eq. (4) can be expressed as a modulation of exponential relaxations,

$$G_{p/q}(t) = \int_0^{\infty} d\eta \int_0^{\infty} d\xi \, \Phi_{p/q}(\eta, \xi) \, e^{-\xi t}, \tag{9}$$

where

$$\Phi_{p/q}(\eta, \xi) = \sum_{l=1}^{n} \sum_{k=1}^{m_l} \frac{b_{l,k}(\zeta_l)}{\pi} \, \eta^{m_l - k} \sin\left(\eta \, \xi^{1/q} \sin(\pi/q) \right) e^{\eta \left(\zeta_l - \cos(\pi/q) \xi^{1/q} \right)}.$$

The rational functions $b_{l,k}(z)$ read

$$b_{l,k}(z) = \frac{d^{k-1}/dz^{k-1} \, \left[(z^q - a)(z^q + a)(z - \zeta_l)^{m_l} / Q(z) \right]}{(m_l - k)! \, (k-1)!},$$

for every $l = 1, \cdots, n$, and $k = 1, \ldots, m_l$. The complex numbers $\zeta_1, \cdots, \zeta_n$ are the roots of the polynomial

$$Q(z) = z^{3q} + z_1 \, z^q + z_{\alpha} \, z^p + z_0 \tag{10}$$

and m_l is the multiplicity of ζ_l, for every $l = 1, \ldots, n$, which means $Q(z) = \Pi_{l=1}^{n} (z - \zeta_l)^{m_l}$ and $\sum_{l=1}^{n} m_l = 3q$.

The case $\alpha = 1/2$ exhibits a simplified exact dynamics described by a finite sum of Eulerian functions:

$$G_{1/2}(t) = \frac{1}{\sqrt{\pi}} \sum_{l=1}^{4} R(z_l) \, z_l \, e^{z_l^2 t} \, \Gamma\left(1/2, z_l^2 t \right), \tag{11}$$

where $R(z)$ is a rational function,

$$R(z) = \frac{(1 - \imath)\left(a^{1/2} + z \right)\left(\imath \, a^{1/2} + z \right)}{2z \left((1 + \imath) \, a + 3a^{1/2} z + 2(1 - \imath) \, z^2 \right)}, \tag{12}$$

while the complex numbers z_1, z_2, z_3 and z_4, are the roots, *distinct* for every positive value of both A and a, of the polynomial $Q(z)$, given by the following form:

$$Q(z) = \pi \sqrt{2/a} \, A + \imath \, a \, z^2 + (1 + \imath) \, a^{1/2} z^3 + z^4. \tag{13}$$

For the sake of shortness, we refer to [8] for a detailed analysis of the case $\alpha = 1/2$ and to [9] for the analytical expressions of the roots. For $\alpha = 3/4$ and $A = a^{9/4} \cos(3\pi/8)/\pi$, the parameter z_1 vanishes and the roots $\zeta_1, \ldots, \zeta_l$ can be evaluated analytically from the solutions of a quartic equation. We do not report the expressions for the sake of shortness. In the remaining cases of rational values of α, the roots of $Q(z)$ must be evaluated numerically, once the numerical values of both A and a are fixed. These details complete the necessary analysis of the function $G_{\alpha}(t)$, driving the exact dynamics.

3. Inverse power laws

The theoretical analysis of the *exact* dynamics, performed above, leads to the following concrete result: a *time scale* τ emerges such that, for $t \gg \tau$, the function $G_\alpha(t)$ exhibits inverse power law behavior described by the asymptotic form

$$G_\alpha(t) \sim -\mathcal{D}_\alpha \, t^{-1-\alpha}, \quad t \to +\infty, \quad 1 > \alpha > 0, \tag{14}$$

where

$$\mathcal{D}_\alpha = \frac{2\,\imath\,\alpha\, a^{2(1-\alpha)} e^{-\imath\pi\alpha/2} \csc\left(\pi\alpha\right) \sec^2\left(\pi\alpha/2\right)}{\pi A\, \Gamma\left(1-\alpha\right)}. \tag{15}$$

A simple choice is

$$\tau_\alpha = \max\left\{1, \left|\frac{3}{z_0}\right|^{1/3}, \left|3\,\frac{z_\alpha}{z_0}\right|^{1/\alpha}, 3\left|\frac{z_1}{z_0}\right|\right\}; \tag{16}$$

the proof is performed in Ref. [7].

Thus, over long timescales, $t \gg \tau_\alpha$, the qubit exact dynamics is described by the asymptotic inverse power law relaxations:

$$\rho_{1,1}(t) = 1 - \rho_{0,0}(t) \sim \rho_{1,1}(0)\ |\mathcal{D}_\alpha|^2\, t^{-2-2\alpha}, \tag{17}$$

$$\rho_{1,0}(t) = \rho^*_{0,1}(t) \sim \rho_{1,0}(0)\, \mathcal{D}_\alpha\, e^{-\imath\omega_0 t}\, t^{-1-\alpha}, \tag{18}$$

for every $\alpha \in (0,1)$.

4. Conclusions

Starting from the initial condition (1) where the reservoir, in the vacuum state, and the qubit are unentangled, the exact dynamics of the qubit interacting, in a rotating wave approximation with a reservoir of bosons described by the spectral density (5) is described by a series (6) of Fox H-functions. Over long timescales, $t \gg \tau_\alpha$, decoherence results in an inverse power law relaxation proportional to $t^{-1-\alpha}$ for every $\alpha \in (0,1)$, according to the choice of the special reservoir (5). The qubit ultimately collapses into the ground state.

An environment implementing the specially designed reservoir of modes can in principle be realized with PBG media [10, 11]. An anisotropic model providing a PBG close to the 3D photonic crystals [14], is discussed in Refs. [12] and [13]. The corresponding density of modes reads

$$\eta\left(\omega\right) \propto \sqrt{\omega - \omega_e}\ \Theta\left(\left(\omega - \omega_e\right)/\omega_e\right),$$

where ω_e is the band edge frequency. If the qubit transition frequency coincides with the edge of the PBG, the low frequency behavior of the specially designed reservoir is recovered by assuming that the couplings vary slowly at low frequencies, $g\left(\omega\right) \simeq g\left(\omega_0\right)$ for $\omega > \omega_0$. Physically, the modes relevant for the dynamics are the resonant ones, which means that the time evolution mostly depends on the frequency behavior near the transition frequency of the system of interest [15]. By placing a qubit in such a material and letting it interact with such a reservoir (in rotating wave approximation), the described decoherence process can in principle emerge. The accuracy of the model depends upon the high frequency behavior of the spectral density.

Also, a structured PBG is the N-period one dimensional lattice discussed in Ref. [16]. It can reproduce a band gap by properly arranging the periodic sequence of unit lattice cells. The density of modes is evaluated analytically as a function of the complex transmission coefficients of each unit cell. Tunable 1D PBG microcavities can potentially supply a realization of structured PBG [17, 18]. Their fabrication is achieved through advanced diffractive grating and photonic crystals technologies. The action of such structured PBG environments on a qubit could be a way of delaying the decoherence process with fundamental applications to Quantum Information Processing Technologies.

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