Many-boson Quantum Walks on Graphs with Shared Coins

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Abstract. Quantum walks of particles obeying Bose statistics are introduced. In such quantum walks the conditional shift operation is performed with a single coin tossing for the whole lattice. An explicit form for the transition probabilities in a single step is derived. This allows to describe the evolution of an arbitrary state and an arbitrary number of steps. This model easily embraces the concepts of the join probability, the counting statistics and the high order correlations. It also presents the computational challenges arising from the exponential increase in the number of basis states entering into the lattice state as a function of the number of quantum walkers and the number of steps. Possible solutions are proposed in some applications of the model to quantum walks on finite graphs.

1. Introduction

After the introduction of quantum walks by Aharonov et al. [1], scores of studies have developed the theory of both discrete-time and continuous-time quantum walks. Quantum walks have revealed strong advantages over their classical counterparts: It has been shown that quantum walks spread quadratically faster compared to their classical counterparts on the line [2, 3]. While the quadratic speed up is also present in search algorithms with discrete quantum walks [4], it is even exponentiated using continuous-time quantum walks in the path searching algorithm over certain types of graphs [5].

Our goal is to make the simplest extension of the single particle quantum random formalism to the case of many indistinguishable particles. Considering an initial state with precisely located walkers, the system evolves to a certain particle distribution. The development of such a formalism could extend our understanding of the concepts of quantum coherence, interference and entanglement. In the following we consider many-particle quantum walks with a shared coin. In other words, we associate a single coin to the whole underlying lattice.

The paper is structured as follows: In the next Section we introduce the mathematical formulation of quantum walks for identical particles. We will use a conventional Hadamard operation, but we will introduce a generalized conditional shift operator. In Section 3 we show how to realize the steps of the random walk and derive a recursion relation between amplitudes. In Section 4 we present the results of simulations of shared coin quantum walks for identical particles. We present results for the particle density distribution, the counting statistics, the Fano-Mandel factor for each vertex occupation number and apply this scheme in the evaluation of the high order correlation coefficients. The last section contains our conclusions and gives a short outlook.

2. 'Graph State'

We consider a $M+1$-vertices's graph. Suppose that this graph is populated by N particles. For example if the graph is the square in Figure 1(b), the initial configuration is

$$\big|\Psi^{[0]}_{\text{GRAPH}}\big\rangle = \frac{1}{\sqrt{2}}\left\{\big|\text{L}\{0;2;1;0\}\big\rangle + \big|\text{R}\{0;2;1;0\}\big\rangle\right\}. \tag{1}$$

where the upper index represent the discrete time step. Taking into account that the number of particles is conserved, the initial 3 particle state on the 4-node graph, will reduce the Hilbert space to 20 possible independent state vectors. $|\text{L}\rangle$ and $|\text{R}\rangle$ are coins chiralities. In general we

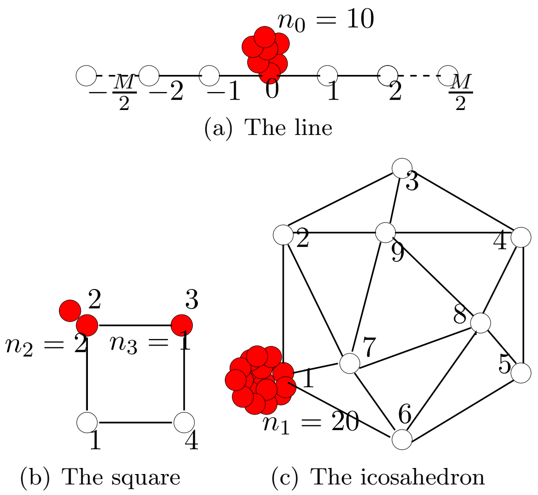


Figure 1. 1(a) the $M+1$-vertices line with $n_0 = 10$ particles, (figure 1(b)) the square graph with $n_2 = 2$ and $n_3 = 1$. 1(c) the icosahedron with $n_1 = 20$ particles.

have

$$\big|\Psi^{[0]}_{\text{GRAPH}}\big\rangle = \sum_{\ell=1}^{20}\sum_{k=1}^{2} \overset{0}{c}_{k\,\ell}\big|v_k\{\overset{0}{n}_{1\,\ell};\overset{0}{n}_{2\,\ell};\overset{0}{n}_{3\,\ell};\overset{0}{n}_{4\,\ell}\}\big\rangle. \tag{2}$$

The coefficients $\overset{0}{c}_{k\,\ell}$s are the weights of configurations. The vertices occupation numbers representation for the system square-graph-3-particles (figure 1(b)) is given by

$$\big|\Psi^{[0]}_{\text{GRAPH}}\big\rangle = \sum_{n_1^0=0}^{3}\sum_{n_2^0=0}^{3}\sum_{n_3^0=0}^{3}\sum_{n_4^0=0}^{3}\sum_{k=1}^{2} C_{\{n_1^0;n_2^0;n_3^0;n_4^0\}}\big|v_k\{n_1^0;n_2^0;n_3^0;n_4^0\}\big\rangle, \tag{3}$$

where the $C_{\{n_1^0;n_2^0;n_3^0;n_4^0\}}$s are products of different types of correlations involved in the selection of occupation numbers over each specific vertex. In general we have

$$\big|\Psi^{[r]}_{\text{GRAPH}}\big\rangle = \sum_{n^r_{-\frac{M}{2}}=0}^{N}\cdots\sum_{n^r_\alpha=0}^{N}\cdots\sum_{n^r_{\frac{M}{2}}=0}^{N}\sum_{k=1}^{d} C_{k\{n^r_{-\frac{M}{2}};\ldots;n^r_\alpha;\ldots;n^r_{\frac{M}{2}}\}}\big|v_k\{n^r_{-\frac{M}{2}};\ldots;n^r_\alpha;\ldots;n^r_{\frac{M}{2}}\}\big\rangle, \tag{4}$$

$$\big|\Psi^{[r]}_{\text{GRAPH}}\big\rangle = \sum_{\ell=1}^{\text{D}}\sum_{k=1}^{d} c^r_{k\,\ell}\big|v_k\{\ldots;n^r_{\alpha\,\ell};\ldots\}\big\rangle \qquad \alpha\in\Big(-\frac{M}{2},\ldots,\frac{M}{2}\Big). \tag{5}$$

The upper index r indicates of the time step of the quantum walks. $\left|\Psi^{[r]}_{\text{GRAPH}}\right\rangle \in \mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_V$; [7], with $\mathcal{H}_C$ the coins Hilbert space and $\mathcal{H}_V$ the position Hilbert space.

3. Generalised conditional shift operator

Each step of the coined quantum walks consists of two operations. The first operation is the coin tossing [8] operation performed using the Hadamard gate

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6}$$

Considering the degrees

$$|\mathrm{L}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\mathrm{R}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

of the coin Hilbert space, the coin tossing respectively transforms $|\mathrm{L}\rangle$ and $|\mathrm{R}\rangle$ as

$$\mathbf{H}|\mathrm{L}\rangle = \frac{1}{\sqrt{2}}(|\mathrm{L}\rangle + |\mathrm{R}\rangle) \text{ and } \mathbf{H}|\mathrm{R}\rangle = \frac{1}{\sqrt{2}}(|\mathrm{L}\rangle - |\mathrm{R}\rangle). \tag{8}$$

The second operation is the particle shifting operation, where the shift is performed using the operator

$$\mathbf{S} = |\mathrm{L}\rangle\langle\mathrm{L}| \otimes \sum_{\eta\in\mathbb{Z}} |\eta-1\rangle\langle\eta| + |\mathrm{R}\rangle\langle\mathrm{R}| \otimes \sum_{\eta\in\mathbb{Z}} |\eta+1\rangle\langle\eta|. \tag{9}$$

In the single particle quantum walk case, the projector $|\eta+1\rangle\langle\eta| \equiv |\eta+1\rangle\langle 0|\cdot|0\rangle\langle\eta|$ corresponds to the product $\hat{\mathbf{a}}^\dagger_{\eta+1}\hat{\mathbf{a}}_\eta$ of quantum creation and annihilation operators. We define $\mathbf{H}_d$, the coin's tossing operator, as

$$\mathbf{H}_d = \left(h_{k\,\jmath}\right) \quad \jmath,\, k \,\in\, (1,2,3,...,d) \text{ and } h_{k\,\jmath} \text{ are roots of the unity;} \tag{10}$$

a d-order Hadamard operator.

$$\mathbf{H}_d|v_k\rangle = \sum_{\jmath=1}^{d} h_{k\,\jmath}|v_\jmath\rangle. \tag{11}$$

The shifting operation combines the graph connectivity and the quantum creation and annihilation operation. Since particles can only be shifted between adjacent vertices, the adjacency matrix intervenes during the shifting operation. In general, it is defined as

$$\mathbf{A} = \left(a_{\mu\,\eta}\right) \quad \mu,\, \eta \,\in\, (1,2,3,...,M+1); \tag{12}$$

$$a_{\mu\,\eta} = \left\langle \begin{array}{l} 1, \text{ if the vertices } \eta \text{ and } \mu \text{ are connected,} \\ 0, \text{ otherwise.} \end{array} \right. \tag{13}$$

For the graph in Figure 1(b), $\mathbf{A}$ is a 4×4 matrix with mirror reflection symmetry such that it splits into

$$\mathbf{A} = \mathbf{A}_{\mathrm{L}} + \mathbf{A}^{\mathrm{T}}_{\mathrm{L}}, \tag{14}$$

$$\text{with} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } \mathbf{A}_{\mathrm{L}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{15}$$

On the other side, let us consider

$$\mathcal{C} = (\hat{\mathbf{a}}_1^\dagger, \hat{\mathbf{a}}_2^\dagger, \hat{\mathbf{a}}_3^\dagger, \hat{\mathbf{a}}_4^\dagger). \tag{16}$$

The $\hat{\mathbf{a}}_\eta$s and $\hat{\mathbf{a}}_\mu^\dagger$s are respectively boson annihilation and creation operators obeying:

$$\big[\hat{\mathbf{a}}_\eta,\, \hat{\mathbf{a}}_\mu^\dagger\big] = \delta_{\eta\,\mu} \;\text{ and }\; \big[\hat{\mathbf{a}}_\eta,\, \hat{\mathbf{a}}_\mu\big] = \big[\hat{\mathbf{a}}_\eta^\dagger,\, \hat{\mathbf{a}}_\mu^\dagger\big] = 0. \tag{17}$$

The conditional shift operator for the square graph Figure 1(b) is given by

$$\mathbf{S} = \mathcal{C}\mathbf{A}_{\mathrm{L}}\mathcal{C}^\dagger \otimes |\mathrm{L}\rangle\langle\mathrm{L}| + \mathcal{C}\mathbf{A}_{\mathrm{L}}^{\mathrm{T}}\mathcal{C}^\dagger \otimes |\mathrm{R}\rangle\langle\mathrm{R}|. \tag{18}$$

The conditional shift operator (18) combined with the Hadamard operator (10) defines the single step quantum walk operator $\mathbf{SH}$. Consider the graph state $\big|\Psi_{\mathrm{GRAPH}}^{[r]}\big\rangle$ (5) at time r. One time step later this state evolves to

$$\big|\Psi_{\mathrm{GRAPH}}^{[r+1]}\big\rangle = (\mathbf{SH})\big|\Psi_{\mathrm{GRAPH}}^{[r]}\big\rangle = \frac{1}{\big\langle\Psi_{\mathrm{GRAPH}}^{[r+1]}\big|\Psi_{\mathrm{GRAPH}}^{[r+1]}\big\rangle}\sum_{s=1}^{\mathrm{D}}\sum_{i=1}^{d} c_{i\,s}^{r+1}\big|v_i\{...;n_{\alpha\,s}^{r+1};...\}\big\rangle, \tag{19}$$

where

$$n_{\alpha\,s}^{r+1} = n_{\alpha\,\ell}^{r} - \delta_{\alpha\,\eta} + \delta_{\alpha\,\mu} \tag{20}$$

and

$$c_{i\,s}^{r+1} = \sum_{\mu=1}^{M+1}\sum_{\eta=1}^{M+1}\sum_{k=1}^{d} a_{\mu\,\eta}^{\mathrm{T}(k)} h_{k\,j} c_{k\,\ell}^{r}\sqrt{(n_{\alpha\,\ell}^{r})^{\delta_{\alpha\,\eta}}(n_{\alpha\,\ell}^{r} - \delta_{\alpha\,\eta} + \delta_{\alpha\,\mu})^{\delta_{\alpha\,\mu}}}. \tag{21}$$

As an example, applying the above to the initial state (2) of the square graph (figure 1(b)), it follows that

$$\begin{aligned}\big|\Psi_{\mathrm{GRAPH}}^{[1]}\big\rangle \;=\; \frac{1}{2\sqrt{5}}\Big(&\big(1+i\big)\sqrt{2}\big|v_1\{1;1;1;0\}\big\rangle + \big(1+i\big)\sqrt{3}\big|v_1\{0;3;0;0\}\big\rangle \\ &+\big(1-i\big)2\big|v_2\{0;1;2;0\}\big\rangle + \big(1-i\big)\big|v_2\{0;2;0;1\}\big\rangle\Big).\end{aligned} \tag{22}$$

4. Results and conclusions

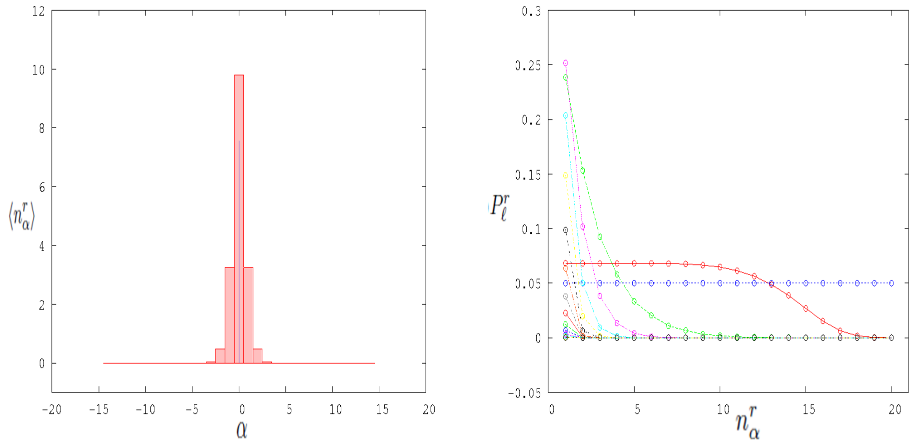
The probability of having a specific configuration at the step r is given by

$$P_\ell^r = \sum_{j=1}^{d} c_{j\,\ell}^{r}\big(c_{j\,\ell}^{r}\big)^*. \tag{23}$$

This probability is the joint probability of the occupation numbers $n_{\alpha\,\ell}^{r}$ over the configuration ℓ. The qth-moment of the occupation number on a vertex α is given by:

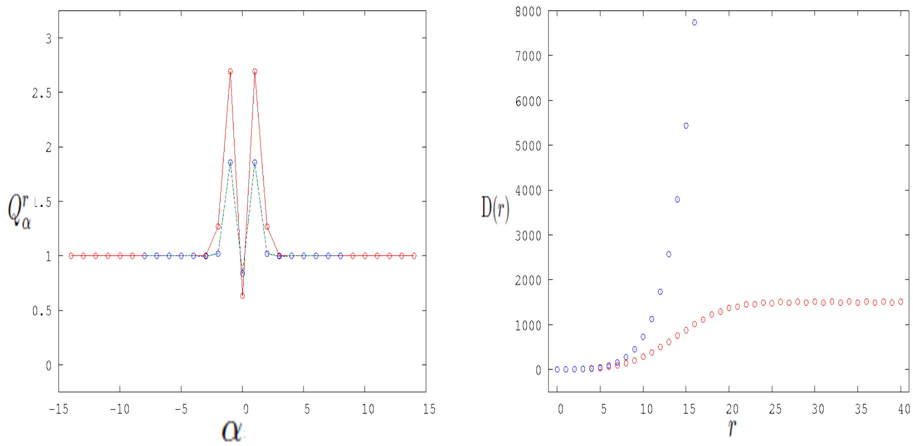
$$\big\langle(n_\alpha^r)^q\big\rangle = \sum_{\ell=1}^{\mathrm{D}} P_\ell^r (n_{\alpha\,\ell}^{r})^q. \tag{24}$$

The counting statistics is directly performed vertex by vertex and step by step. The shared coin's many-particle quantum walks permit the direct accessibility to all elements of the basis entering in the formation of the graph state. We record the number of occupants as they enter



(a) The particle distribution after 28 steps (b) The vertices counting statistics

Figure 2. (b) The particle density distribution the solid blue line is the initial distribution and the reds bars are the distribution after 28 steps. (b) The vertices wise counting statistics for the shared coins many-particle quantum walks for the coherent state $|z\rangle = \frac{1}{N}\sum_{n=0}^{20} \frac{z^n}{\sqrt{n!}}|n\rangle$. The horizontal blue line represents the counting at step $r = 0$ and all others are after $r = 28$. The particles counting statistics on vertex $\alpha = 0$ shows an increase in the counting of $n_\alpha^r \leq 13$.



(a) The Fano-Mandel factor Q_α^r. (b) The number D(r) of configurations

Figure 3. (a) The vertices's Fano Q_α^r factor for 16 steps (blue lines) and 28 steps (red color). (b) Comparison between the numbers D(r) of the configurations effectively involved in the formation of the graph state $\left|\Psi_{\text{GRAPH}}^{[r]}\right\rangle$. The blue dots are for the line while the reds are a 20 vertices cycle both with 20 quantum walkers.

in the formation of the occupation numbers and the frequencies of the different occupation numbers.

The Fano-Mandel factor Q_α^r defines the deviation of the counting statistics from Poissonian statistics. It is given by

$$Q_\alpha^r = \frac{\langle (n_\alpha^r)^2 \rangle - \langle n_\alpha^r \rangle^2}{\langle n_\alpha^r \rangle}. \tag{25}$$

If $Q_\alpha^r = 1$, the vertex α counting statistics corresponds to the Poisson distribution. For $Q_\alpha^r < 1$, the vertex α counting statistics corresponds to sub-Poissonian distribution which indicates particles bunching on the vertex. And $Q^r_\alpha > 1$ the vertex α counting statistics corresponds to super-Poissonian distribution which indicates particles anti-bunching on the vertex.

The study of many-particle quantum walks faces the challenge of dealing with an exponential increase in the number of parameters needed for the control of the walkers. In the shared coin many-particle quantum walks, this problem is related to the number $\mathrm{D}(r)$ of configurations effectively entering in the formation of the graph state. So far we have recorded attempts with 8 quantum walkers in continuous or discrete quantum walks. In the discrete many-particle quantum walks the highest number of steps is around $50 = 2 \times 25$ [10], where the study is performed over 8 quantum walkers. In general for an open type of graphs $\mathrm{D}(r)$ increases exponentially (figure 3(b)). When we consider finite graphs the expansion slows down significantly and therefore this model can be used in the study of many-particle quantum walks on finite graphs. In the case finite graphs the number $\mathrm{D}(r)$ converges to an upper bound (figure 3(b)) after a certain number of steps.

In conclusion, we have presented a simple generalization to many particles of a quantum random walk. The method is based on the use of a shared coin. Although, such shared coin many-particle quantum walks are computationally challenging, quite efficient ways of performing the counting statistics are available.

Acknowledgments
This work is based on research supported by the South African Research Chair Initiative of the Department of Science and Technology and National Research Foundation.

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