Quasi-normal Modes for Spin-3/2 Fields

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Abstract. With the Large Hadron Collider already able to produce collisions with an energy of 8 TeV, the formation of higher dimensional black holes may soon be possible. In order to determine if we are detecting these higher dimensional black holes we need to have a theoretical understanding of what the signatures of such black holes could be. As such we shall discuss quasi-normal modes (QNMs) for spin-3/2 fields as they travel through a black hole background. We will begin by studying possible QNMs for N-dimensional Schwarzschild black holes, we will then calculate allowed QNMs for N-dimensional Reissner-Nordström black holes. We will use the Wentzel-Kramers-Brillouin approximation to determine the QNMs for the two types of black holes described above.

1. Introduction

There are currently many ways to indirectly detect black holes [1]. One possible direct way of detecting a black hole is through gravitational wave detection. These are oscillations in the space time and are emitted by black holes when they are perturbed through either black hole-black hole collisions, stellar collapse or other matter interactions. These perturbations cause non-radial oscillations on the surface of a black hole and are called quasi-normal modes (QNMs). These QNMs in turn effect the space-time around a black hole and result in quasi-normal frequencies being emitted from black holes, these frequencies would be detected as gravitational waves, or particles [2]. What is interesting is that the allowed quasi-normal frequencies of a black hole are directly related to the characteristics of the black holes. For instance, the allowed quasi-normal frequencies of a Schwarzschild black hole are given by the mass of the black hole, but for a Kerr-Newman black hole they are determined by the angular momentum, charge and mass of the black hole [2]. This means that we could not only directly detect black holes by detecting their gravitational emission, but we could also get an idea of the parameters of the black hole by using this. A more elaborate discussion on detecting these QNMs is given in reference [3]. In recent years the study of higher dimensional general relativity, and by extension the study of higher dimensional black holes, has become a subject of increasing interest. One of the reasons for this interest is that string theory requires more than four dimensions in order to properly account for gravity, in fact in string theory a 5-dimensional black hole is required to correctly account for the entropy of a black hole [4]. Another reason for the study of higher dimensional black holes is that Anti de-Sitter/Conformal field theory (AdS/CFT) correspondence tells us that the dynamics of an N-dimensional black hole correspond to those of an $N-1$ dimensional quantum field theory [5]. A more elaborate discussion on the reasons for studying N-dimensional black holes is given in reference [6]. In order to calculate the allowed QNMs for the various types of black holes we will use methods such as the Wentzel-Kramers-Brillouin (WKB) method [7] and the improved asymptotic iteration method (AIM) [8]. We shall first give a brief overview of black holes and how to mathematically construct N-dimensional black holes. We then give a brief discussion of QNMs and then give an overview of the WKB method and improved AIM. We will provide a brief overview of what we have done so far and give examples of possible future work.

2. Black Holes

Black holes have been described as the hydrogen atom of general relativity, since they are relatively simple general relativistic objects which display all of the properties predicted by Einstein's field equations. Hence a better understanding of black holes would allow us to better understand general relativity, just as the hydrogen atom can help in our understanding of quantum mechanics. In this paper we will set c = 1.

2.1. Schwarzchild Black holes

The Schwarzschild metric can be used to describe the space time surrounding any non-rotating objects in a vacuum [9]. It is given as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dx^2 + r^2 \left(d\theta^2 + \sin(\theta) d\phi^2\right), \tag{1}$$

where $G = 7.426 \times 10^{-28} m/kg$, the mass (M) denotes the mass of the object and r denotes the radial distance between a point in the space time and the center of that object. This metric is valid for all values of r greater than zero. We can describe the space time surrounding a black hole by looking at systems for which the value of $2GM$ is large compared to r. When r approaches $2GM$ our metric will produce a singularity, where this singularity is in fact a "coordinate singularity"; which means that there exists a problem with the coordinate system we have chosen rather than there being any actual physical singularity at that point in space-time [9]. The point $r = 2GM$ is in fact the radial location of the event horizon of the black hole. These types of black holes are the simplest types of black holes and provide the simplest case for studying QNMs.

2.2. Reissner-Nordström black holes

Reissner-Nordström black holes are a more general case of the Schwarszchild black hole, that is, they are non-rotating electrically charged black holes. Hence the metric is similar to that of the Schwarszchild black hole, and is given as follows [10]:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin(\theta) d\phi^2\right), \tag{2}$$

where G, M and r represent the same quantities as they did for the Schwarzchild metric. Q represents the electric charge of the black hole.

2.3. N-dimensional metrics

In the above metrics we have temporal, radial and spherical components, denoted by dt^2, dr^2 and $d\theta^2$ and $d\phi^2$ respectively. The spherical components of these 4-dimensional metrics are given by the 2-sphere metric. We can rewrite the metrics as follows:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2, \tag{3}$$

where $f(r)$ denotes the radial functions associated to the temporal and radial components, and $d\Omega_2^2$ is the metric for the 2-sphere. For N-dimensional black holes we can rewrite the metric as follows

$$ds_N^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{N-2}^2, \tag{4}$$

where $f(r) = 1 - 2GM/(r)^{N-3}$, for the case of an N-dimensional Schwazrchild black hole, or $f(r) = 1 - 2GM/(r)^{N-3} + Q^2/(r)^{2N-6}$, for the case of an N-dimensional Reissner-Nordström black hole, and where $d\Omega_N^2 = d\theta^2 + \sin(\theta)^2 d\Omega_{N-1}$ with $d\Omega_{N-1}$ the metric for the $N-1$-sphere [11, 12, 13].

3. Quasi-Normal Modes

As stated before, QNMs are the reaction to a black hole being perturbed in some way. A simple explanation of what a QNM is can be described by the tapping of a wine glass. If we tap the wine glass we are in fact perturbing the glass. This perturbation causes the glass to resonate at specific frequencies. The allowed frequencies are determined by the composition of the glass, and even the contents within the glass. If the glass continued to resonate forever then we would be generating a normal mode. Since the glass does stop resonating we say it is a QNM. So when studying QNMs we are studying standing waves with some damping term attached. If we consider our wine glass example we should note that wine glasses with different amounts of water in them ring at different frequencies, so we could, in theory, determine the amount of water in a glass by the sound it makes when we tap the glass. A similar idea can be used for the emitted quasi-normal frequencies of a black hole. Although with a black hole energy is being radiated, not as sound, but rather as disruptions in the space time which we call gravitational waves.

3.1. A Mathematical description of QNMs

In order to develop a more mathematical understanding of QNMs we can look at the mathematical formula for standing waves since, as stated above, QNMs are merely damped standing waves. So we can represent these QNMs as follows [14]:

$$\frac{d^2}{dx^2}\Psi - \frac{d^2}{dt^2}\Psi - V(x)\Psi = 0. \tag{5}$$

With Ψ describing some wavelike function where x and t denote space and time coordinates respectively. $V(x)$ is some x-dependent potential, the 4-dimensional potential is presented in Eq.(16). We can solve the equation as we would a standing wave problem, and so we can assume the following time dependency [3]:

$$\Psi(x,t) = e^{-i\omega t}\phi(x), \tag{6}$$

where ω is the frequency of the wave and is proportional to the energy eigenvalue. Plugging this into our equation for a QNM we get:

$$\frac{d^2\phi(x)}{dx^2} - (\omega^2 + V(x))\phi(x) = 0. \tag{7}$$

This is the general form for the QNMs we will be studying in this paper. In order to solve this equation we require specific boundary terms, similar to those required for solving the standing wave equation. The boundary conditions that we will impose are:

$$\begin{aligned} &V \to 0; x \to \infty, \\ &V \to 0; x \to -\infty. \end{aligned} \tag{8}$$

This means that particles fall into the black hole at the horizon, and those which move to infinity are no longer influenced by the black hole. In order to determine if a frequency of a field is in fact a QNM we need to ensure that it obeys these requirements. In this project we will use both the WKB method, to 6th order [7], and the improved AIM [8] to determine the solutions to our QNM equation.

4. Approximation Methods

4.1. WKB Approximation

The WKB method is a well known approximation, and is usually used in quantum mechanics to determine the solutions of Schrödinger equations in systems with non-constant potentials. The WKB method can be used to determine the approximate solution to second order differential equations. A brief example of how the method is used to solve second order differential equations is given below. We can use the WKB approximation to solve problems of the following form [15]:

$$\epsilon^2 \frac{d^2 y}{dx^2} = Q(x) y, \tag{9}$$

where $\epsilon \ll 1$ and positive. Using the WKB method we get a solution of the following form,

$$\begin{aligned} &\epsilon^2 \left(\left(A_0''(x) + \epsilon A_1''(x) + ... \right) + 2 \left(A_0'(x) + \epsilon A_1'(x) + ... \right) \frac{iu'(x)}{\epsilon} \right) + \\ &\epsilon^2 \left[\left(A_0(x) + \epsilon A_1(x) + ... \right) \left(\frac{iu''(x)}{\epsilon} - \left(\frac{u'(x)}{\epsilon} \right)^2 - Q(x) \left(A_0(x) + \epsilon A_1(x) + ... \right) \right) \right] = 0. \end{aligned} \tag{10}$$

Where $A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + ...$ is a series expansion of $A(x, \epsilon)$ with primes denoting derivatives with respect to x. [16]. We obtain solutions for $A_0(x), A_1(x), A_2(x), ...$ by grouping the term by powers of ϵ.

We can use the WKB method to study black hole perturbations since these equations reduce to the form of a wave equation, similar to the example above [3]. This equation, with the known potential, is then solved using the the WKB approximation. The cases for Schwarzschild black holes and Reissner-Nordström black holes has been studied extensively [8], however these studies have not considered the case of spin-3/2 fields.

4.2. Improved AIM

The theory of this method is given in reference [17]. The theorem of this method is given below: Given λ_0 and s_0 in $C_\infty(a, b)$, then the differential

$$y'' = \lambda_0(x) y' + s_0(x) y, \tag{11}$$

has the general solution of:

$$y(x) = exp\left(- \int^x \alpha dt \right) \left[C_2 + C_1 \int^x exp\left(\int^t \left(\lambda_0(\tau) + 2\alpha(\tau) \right) d\tau \right) \right], \tag{12}$$

for some $n > 0$.

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha, \tag{13}$$

where $\lambda_k = \lambda'_{k-1} + s_{k-1} + \lambda_0 \lambda_{k-1}$ and $s_k = s'_{k-1} + s_0 \lambda_{k-1}$ for $k = 1, 2, ..., n$. We again solve for the QNMs by first obtaining the potential energy term for a spin-3/2 particle in a black hole background, and then use the AIM approximation to obtain the allowed QNMs.

5. Rarita-Schwinger fields in 4D Schwarzschild background

For this case we have only considered massless Rarita-Schwinger fields,spin-3/2 particles, where to determine how these fields propagate through space time we used the Rarita-Schwinger equation, given as [18],

$$\gamma^{\mu\nu\alpha}\nabla_\nu\Psi_\alpha = 0, \tag{14}$$

where,

$$\gamma^{\mu\nu\alpha} = \gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\mu g^{\nu\alpha} + \gamma^\nu g^{\mu\alpha} - \gamma^\alpha g^{\mu\nu}. \tag{15}$$

Using this equation we can determine the equations of motion for a massless Rarita-Schwinger field. Which in turn gives us the potential function for these particles given as [18]

$$V_{1,2} = \pm f(r)\frac{dW}{dr} + W^2, \tag{16}$$

with,

$$W = \frac{\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right)\sqrt{f(r)}}{r\left(\left(j+\frac{1}{2}\right)^2 - f(r)\right)}, \tag{17}$$

$f(r) = ((r-2M)/r)$ and $j = 3/2, 5/2, 7/2, ...$

In Table 1 we present our results for the 4-dimensional Schwarzschild black hole, using the 3rd order WKB approximation as given in reference [19], and 6th order as given in reference [7]. We compare this to results obtain using the AIM method as given in reference [17]. The QNMs calculated agree across the methods, within their numerical uncertainties, to those obtained in reference [20].

Table 1. Low-lying ($n \leq l$, with $l = j - 3/2$) gravitino quasinormal mode frequencies using the WKB and the AIM methods [18].

		WKB		AIM
l	n	3rd Order	6th Order	150 iterations
0	0	0.3087 - 0.0902i	0.3113 - 0.0902i	0.3108 - 0.0899i
1	0	0.5295 - 0.0938i	0.5300 - 0.0938i	0.5301 - 0.0937i
1	1	0.5103 - 0.2858i	0.5114 - 0.2854i	0.5119 - 0.2863i
2	0	0.7346 - 0.0949i	0.7348 - 0.0949i	0.7348 - 0.0949i
2	1	0.7206 - 0.2870i	0.7210 - 0.2869i	0.7211 - 0.2871i
2	2	0.6960 - 0.4844i	0.6953 - 0.4855i	0.6892 - 0.4834i
3	0	0.9343 - 0.0954i	0.9344 - 0.0954i	0.9344 - 0.0954i
3	1	0.9233 - 0.2876i	0.9235 - 0.2876i	0.9235 - 0.2876i
3	2	0.9031 - 0.4835i	0.9026 - 0.4840i	0.9026 - 0.4840i
3	3	0.8759 - 0.6835i	0.8733 - 0.6870i	0.8733 - 0.6870i
4	0	1.1315 - 0.0956i	1.1315 - 0.0956i	1.1315 - 0.0956i
4	1	1.1224 - 0.2879i	1.1225 - 0.2879i	1.1225 - 0.2879i
4	2	1.1053 - 0.4828i	1.1050 - 0.4831i	1.1050 - 0.4831i
4	3	1.0817 - 0.6812i	1.0798 - 0.6830i	1.0798 - 0.6830i
4	4	1.0530 - 0.8828i	1.0485 - 0.8891i	1.0485 - 0.8891i

6. Concluding remarks

QNMs are very sensitive to the initial conditions which created them, therefore is it necessary to calculate them using very precise methods. We have done this by using the WKB method to 6th order and taking 150 iteration of the AIM.

As stated previously the real part of our QNMs represents their frequencies, which is proportional to their energies, and the imaginary parts represent how damped our QNMs are. In Table 1 we can clearly see that by increasing the angular quantum number, l, we increase the frequency, and therefore energy, of our emitted QNMs. Note also that an increase in the mode number, n, results in a decrease of the frequency. We also see that an increase in l results in an increase in the strength of the damping associated to the QNMs. This means that lower n modes are more unlikely to be detected compared to the higher n modes, with the emission spectra being comprised predominately of our more energetic QNMs, with smaller dampening terms.

Since we have shown that our method can reproduce the values for the 4-dimensional case, we are working on a solution for the N-dimensional Schwarzschild metric and will then investigate the allowed QNMs for Rarita-Schwinger fields in a Reissner-Nordstöm black hole background, both in four dimensions and the N-dimensional case. We would then like to investigate black holes in AdS space times, as then we could begin investigating the connection between N-dimensional black holes and the dynamics of $N-1$ CFT, where a full review of this is given in reference [5].

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