Minimum Norm Estimates for Bioelectromagnetic Inverse Problems

Neuroimaging

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Abstract

Electroencephalography (EEG) is a non-invasive tool used to reconstruct sources of cerebral activity generated in the human brain. The sources of cerebral activity are modelled using a current density vector. The inverse problem considered attempts to reconstruct the current density vector, given a model of the head (which determines a Lead Field Matrix) and a set of recorded scalp potentials. The inverse problem is under determined and hence no unique solution is possible. In this study uniqueness is achieved by constructing the minimum norm solution.

Introduction

The EEG forward problem describes the scalp potentials sourced by brain activity. The primary current vector describes the current sourced by the brain. The primary currents occur due to movements of ions within the dendrites of pyramidal cells in the active regions of the brain. A stimulus will excite many excitary synapses of a whole pattern of neurons which leads to a negative current just under the brain surface and a positive current quite close but underneath. The source is thus a "dipole current" modelled as

$$\vec{J}^p = hI\vec{e}_d\delta(\vec{x}-\vec{x}_0) \tag{1}$$

h is the distance between the source/sink separation, I is the magnitude of the current and $\vec{e}_d$ points from the source to sink, i.e. parallel to flow of positive current.

The relevant physics is captured in Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \tag{2}$$

We will also make use of the continuity equation

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \tag{3}$$

Bioelectromagnetism deal with frequencies ≤ 100Hz, so that we can use the quasistatic approximation. The current density in a passive non-magnetic medium can be divided into an Ohmic current and a polarization current

$$\vec{J} = \sigma\vec{E} + \frac{\partial\vec{P}}{\partial t} \tag{4}$$

where $\vec{P} = (\epsilon-\epsilon_0)\vec{E}$ with ϵ the electrical permittivity of the medium. In general, σ is a 3×3 matrix σ that is symmetric and has positive eigenvalues. Different points in our medium can have a different conductivity.

In the quasistatic approximation $\dot{\vec{B}} = 0$ so that $\vec{\nabla}\times\vec{E} = 0$ and hence that $\vec{E}$ is the gradient of a potential. Further, because $\dot{\vec{E}} = 0$ the polarization current vanishes. Since the brain is an active medium we can write

$$\vec{J} = \vec{J}^p - \sigma\vec{\nabla}V \tag{5}$$

where $\vec{J}^p$ is the active source of current from brain activity and $-\sigma\vec{\nabla}V$ is the surviving term in the quasistatic approximation from (4). Now

$$\rho = \epsilon_0\vec{\nabla}\cdot\vec{E} \qquad \Longrightarrow \qquad \frac{\partial\rho}{\partial t} = \epsilon_0\vec{\nabla}\cdot\dot{\vec{E}} = 0 \tag{6}$$

so that the continuity equation implies that $\vec{\nabla}\cdot\vec{J} = 0$. This then implies that

$$\vec{\nabla}\cdot(\sigma\vec{\nabla}V) = \vec{\nabla}\cdot\vec{J}^p \tag{7}$$

This is the potential equation for the EEG forward problem. It describes the electric potential V in the head due to the primary current $\vec{J}^p$ caused by brain activity. The forward problem is defined on domain Ω.

Head models have different compartments for the scalp, skull, brain tissue etc. The conductivity σ jumps as we move between the compartments. We can argue that V is continuous across compartments. Since charge does not build up anywhere the integral of $\vec{\nabla}\cdot\vec{J}^p$ over a box straddling a boundary between two compartments will vanish. This implies that $\sigma\hat{n}\cdot\vec{\nabla}V$ is continuous across compartment walls. Since σ itself is not, we learn that $\vec{\nabla}V$ has to jump across compartment boundaries and hence that $\vec{\nabla}V$ is not well defined on the boundary of a compartment. Finally, since $\sigma = 0$ for the air surrounding the head we learn that

$$\hat{n}\cdot\sigma\vec{\nabla}V = 0 \qquad \text{on} \qquad \partial\Omega \tag{8}$$

The potential equation (7) and boundary condition (8) together constitute the EEG forward problem.

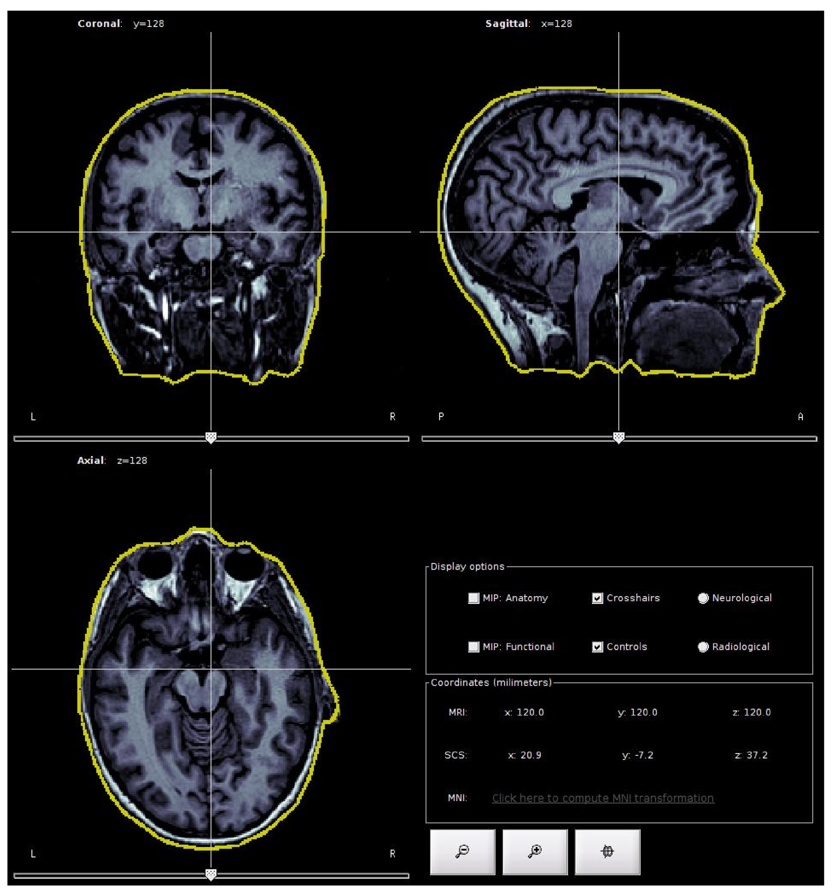

Figure 1: An example of a head model generated using the MATLAB neuroimaging toolbox, Brainstrorm. This is a crude model with only two compartments - and outer skull and an inner skull.

Lead Vector and Lead Field

Lead field theory is important for many inverse source localization algorithms. After computing the lead field matrix the inverse problem can be recast as a finite dimensional linear problem. The lead vector and lead field take advantage of the linearity of electromagnetism.

The *lead vector* describes how 3 dipoles with unit strength, parallel to the three Cartesian directions at a fixed location $\vec{x}_0$, set up a potential at the surface for a given pair of EEG leads. To build the lead vector start from the potential $U_x(\vec{p})$ which describes the potential at $\vec{p}$ set up by a unit dipole at $\vec{x}_0$, oriented along the $\hat{x}$ direction. Similarly the potential $U_y(\vec{p})$ describes the potential at $\vec{p}$ set up by a unit dipole at $\vec{x}_0$, oriented along the $\hat{y}$ direction and the potential $U_z(\vec{p})$ describes the potential at $\vec{p}$ set up by a unit dipole at $\vec{x}_0$, oriented along the $\hat{z}$ direction. A general dipole has a dipole moment that can be written as

$$\vec{d} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} = (r_x, r_y, r_z) \tag{9}$$

Arranging the potentials above into a vector $\vec{c} = (U_x(\vec{p}), U_y(\vec{p}), U_z(\vec{p}))$, we can write the potential set up by this $\vec{d}$ dipole as

$$U(\vec{p}) = r_xU_x(\vec{p}) + r_yU_y(\vec{p}) + r_zU_z(\vec{p}) = \vec{c}(\vec{p})\cdot\vec{d} \tag{10}$$

Now, consider a pair of EEG electrodes located at $\vec{a}$ and $\vec{b}$. We have

$$U(\vec{a}) = \vec{c}(\vec{a})\cdot\vec{d} \qquad U(\vec{b}) = \vec{c}(\vec{b})\cdot\vec{d} \tag{11}$$

for the potential at these two points sourced by the dipole. Thus, the potential difference is

$$V_{\vec{a}\vec{b}} = U(\vec{a}) - U(\vec{b}) = \vec{r}\cdot(\vec{c}(\vec{a}) - c(\vec{b})) \equiv \vec{r}\cdot\vec{I}_{\vec{a}\vec{b}} \tag{12}$$

A pair of surface electrodes is called a *lead*. We say that $\vec{I}_{\vec{a}\vec{b}}$ is the *lead vector* for the lead positioned at $\vec{a}$ and $\vec{b}$.

Lead Field and Lead Field Matrix

If a dipoles position is changed, the lead vector will change $\vec{I}_{\vec{a}\vec{b}}$. If the orientation or strength of the dipole is changed, the lead vector will not change but, of course, $\vec{r}$ will change. Thus, we should write the lead vector as $\vec{I}_{\vec{a}\vec{b}}(\vec{x}_0)$. Computing the lead vector for each possible dipole location $\vec{x}_0$ within the volume conductor will give a vector field. This vector field is called the *lead field*.

Consider a set of S points on the scalp, described by $\vec{x}_a$ with $a = 1, 2, ..., S$. Choose $\vec{x}_S$ as a reference. We then get $S-1$ lead vectors

$$\vec{I}_a(\vec{x}_0) \equiv \vec{I}_{\vec{x}_a\vec{x}_S}(\vec{x}_0) \qquad a = 1, 2, \cdots, S-1 \tag{13}$$

We can build an $S-1$ dimensional vector of potential differences across the $S-1$ leads as follows

$$\begin{bmatrix} \vec{I}_1(\vec{x}_0)\cdot\vec{r} \\ \vec{I}_2(\vec{x}_0)\cdot\vec{r} \\ \vdots \\ \vec{I}_{S-1}(\vec{x}_0)\cdot\vec{r} \end{bmatrix} \equiv L(\vec{x}_0)\cdot\vec{r} \tag{14}$$

$L(\vec{x}_0)$ is a matrix valued field, the lead field matrix. As a matrix $L(\vec{x}_0)$ is $(S-1)\times 3$ dimensional.

Our last generalization is to write the above vector of potential differences for the case that we have m active dipoles at locations $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_m$ each with its own strength $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_m$. In this case we assemble the strengths into a $3m$ dimensional strength vector and we obtain a lead field matrix that is $(S-1)\times 3m$ dimensional.

Full Subtraction Approach

This approach is motivated by the following two observations: Firstly, we can solve for the potential set up by a dipole source in an infinite homogeneous conductor. The solution is singular at the source. Secondly, it is difficult to numerically solve for the potential set up by dipole sources, in the head, a non-homogeneous medium. The source of the difficulty is singular potential at the source.

We can use the analytic solution close to the source and match this to a numerical solution. In this way, the only piece of the solution we need to construct numerically is removed from source and consequently, not singular. The full subtraction approach is simply a way of doing this: Break up the domain of the head (Ω) into a region (Ω^∞) which surrounds the source and is small enough that the conductivity is homogeneous in this small domain. Denote the constant value of the conductivity on this small domain by σ^{source}. Introduce a conductivity $\sigma^\infty(x) = \sigma^{\text{source}}$ $x\in\Omega$. i.e. this conductivity is constant over the whole domain Ω. The actual conductivity of the problem σ can now be written as

$$\sigma(x) = \sigma^\infty(x) + \sigma^c(x) \qquad x\in\Omega \tag{15}$$

This last formula is our definition of $\sigma^c(x)$. Note that $\sigma^c(x) = 0$ $x\in\Omega^\infty$. Break the potential into two pieces $V = V^\infty + V^c$. V^∞ is the piece of the potential that we will solve for analytically; it is singular on Ω^∞ at the location of the source. V^c is the piece of the potential that we will solve for numerically; it is regular throughout Ω. To derive the analytic formula for V^∞ we will make a further simplifying assumption that σ^{source} is well described by a homogeneous isotropic conductivity, that is $\sigma^{\text{source}} = \sigma_0\mathbf{1}$ where σ_0 is a constant and $\mathbf{1}$ is the 3×3 identity matrix. Our source is a dipole located at $\vec{x}_0\in\Omega^\infty$ and it has dipole moment $\vec{m}$. V^∞ is obtained by solving Poisson's equation

$$\vec{\nabla}\cdot\vec{\nabla}V^\infty(\vec{x}) = \frac{\vec{m}\cdot\vec{\nabla}\delta(\vec{x}-\vec{x}_0)}{\sigma_0} \tag{16}$$

with boundary condition $V^\infty(\vec{x})\to 0$ as $|\vec{x}|\to\infty$. The solution is

$$V^\infty(\vec{x}) = \frac{1}{4\pi\sigma_0}\frac{\vec{m}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} \tag{17}$$

The full Poisson equation

$$\vec{\nabla}\cdot(\sigma\vec{\nabla}V) = \vec{\nabla}\cdot\vec{J}^p \tag{18}$$

now implies the following equation for V^c

$$-\vec{\nabla}\cdot([\sigma^\infty + \sigma^c]\vec{\nabla}V^c) = \vec{\nabla}\cdot(\sigma^c\vec{\nabla}V^\infty) \tag{19}$$

with the boundary condition

$$\hat{n}\cdot\sigma\vec{\nabla}(V^\infty + V^c) = 0 \qquad \text{on} \qquad \partial\Omega \tag{20}$$

To obtain a numerical implementation, expand the potential in terms of some finite element basis functions $\phi_j(\vec{x})$ as

$$V^c(\vec{x}) = \sum_{j=1}^N \phi_j(\vec{x})V_j \tag{21}$$

To get the numerical solution we need to obtain the V_j by solving the following linear system

$$\sum_{i=1}^N K_{ji}V_i = b_j \tag{22}$$

where

$$\begin{aligned} K_{ij} &= \int_\Omega \vec{\nabla}\phi_j\cdot\sigma(\vec{x})\cdot\vec{\nabla}\phi_i d^3x \\ b_i &= \int_\Omega \vec{\nabla}\phi_i\cdot(\sigma(\vec{x}_0) - \sigma(x))\cdot\vec{\nabla}V^\infty(\vec{x})d^3x \\ &\quad - \int_{\partial\Omega}\vec{n}\cdot\sigma(\vec{x}_0)\cdot\vec{\nabla}V^\infty(\vec{x})\phi_i(\vec{x})d^2x \end{aligned} \tag{23}$$

Results

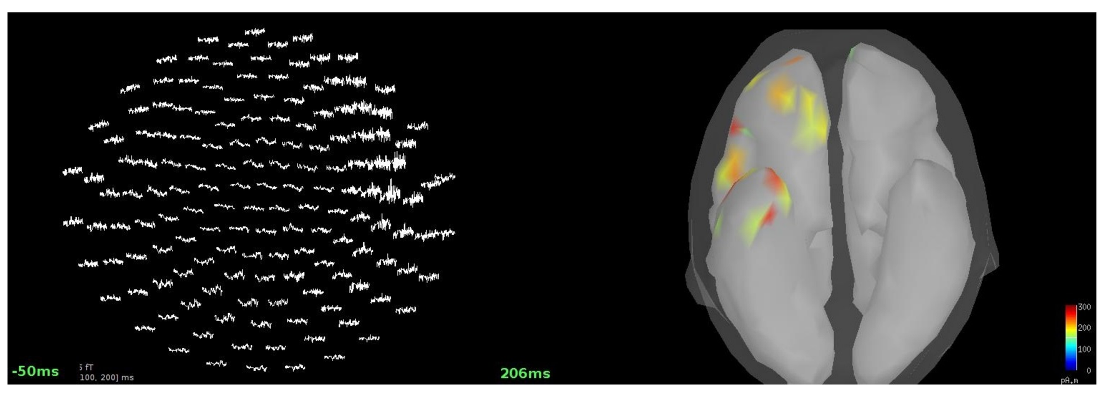

Figure 2: In the left figure above, the 2d layout of the reading of thje scalp potentials is show. The rightmost figure above shows the reconstructed source of the activation, on the cortex.

References

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