Vacuum energies and frequency dependent interactions

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Abstract. The vacuum polarization (or Casimir) energies of field configurations whose interaction with the quantum fluctuations is frequency independent are straightforwardly computed from scattering data since there is a simple relation between the frequency derivative of the scattering phase shift and a spatial integral of the Green's function at coincident points. In more complicated, but nevertheless typical frameworks, the interaction of the quantum fluctuations is frequency dependent and this relation must be modified. This modification may or may not additionally contribute to the vacuum polarization energy. Here we will consider three examples that naturally induce frequency dependent interactions. (I) Scalar electrodynamics with a static background potential. (II) An effective theory that emerges from integrating out a heavy degree of freedom. (III) Quantum electrodynamics coupled to a frequency dependent dielectric material. In cases (II) and (III) any omission of the frequency dependence violates the renormalizability of the theory. For case (III) an ambiguity arises arises because the introduction of a dielectric function comes at the expense of lacking a canonical Lagrangian formulation for the interaction of the photons with the constituents of the material. The physically motivated choice for the Hamiltonian leads to an attractive self–stress of a dielectric sphere.

1. Introduction

In quantum field theory the vacuum polarization (sometimes called Casimir[1]) energy refers to the change of the zero point energies due to the interaction of the quantum fluctuations with a (static) background $V(\mathbf{x})$. This background can be generated by a localized solution to the classical field equations, or it can model an interaction with some material, or it can mimic boundary conditions on the quantum fluctuations in some singular limit[2].

The energy eigenvalues emerge from a Schrödinger type wave–equation

$$\left[-\nabla^2 + V(\mathbf{x})\right]\psi_\nu(\mathbf{x}) = k_\nu^2\psi_\nu(\mathbf{x})\,, \tag{1}$$

with $\omega_\nu = \frac{k_\nu^2}{2m}$ or $\omega_\nu = \sqrt{k_\nu^2+m^2}$ for non–relativistic or relativistic dispersion relations, respectively. Here m is the mass of the quantum field and natural units ($\hbar = 1$, $c = 1$) are adapted. Let $\omega_\nu^{(0)}$ be the energy eigenvalues of the non–interacting problem, *i.e.* when $V \equiv 0$. Then the vacuum polarization energy is

$$E_{\rm vac} = \frac{1}{2}\sum_\nu \left[\omega_\nu - \omega_\nu^{(0)}\right] + E_{\rm ct}\,, \tag{2}$$

where $E_{\rm ct}$ is the counterterm contribution that cancels the ultra–violet divergence in the sum unambiguously. This equation is not suitable for calculations. First we have to distinguish between bound state and (continuum) scattering solutions to the wave–equation (1). The bound states energies are simply ω_n while the scattering solutions, whose energies are given by a continuous function $\omega(k)$, are characterized by the S–matrix or the phase shift $\delta(k)$ for a single mode[1]. Properly identifying these contributions leads to the phase shift formula

$$E_{\rm vac} = \frac{1}{2}\sum_{n={\rm b.s.}} |\omega_n| + \int \frac{dk}{2\pi}\,\omega(k)\,\frac{\partial}{\partial k}\delta(k)\,.+E_{\rm ct} \qquad (3)$$

This formula can be derived/motivated in various ways. For example, it is popular to express the continuum contribution as the integral over frequencies weighted by change of the density of states induced by the background. This change is computed by the derivative of the phase shift according to the Krein–Friedel–Lloyd (KFL) formula; see Ref.[3]. However, there is a more rigorous approach based on the vacuum matrix element of the energy momentum tensor and Green's function methods[4]. In that calculation the frequency derivative of the wave–equation (1)

$$\frac{\partial}{\partial k}\delta(k) = \int dx\, \mathsf{Im}\left\{G(x,x,k)\frac{\partial}{\partial k}\left[k^2 - V(\mathbf{x})\right] - 2kG^{(0)}(x,x,k)\right\}\,, \qquad (4)$$

proves essential. Here $G(x,y,k)$ is the Green's function in the considered channel. It is hence very suggestive that a more complicated interaction with the background, that in particular is frequency dependent, will give rise to modifications of the phase shift formula (3). The main purpose of this study is to investigate possibilities for such modifications.

Before doing so, we must rephrase (3) in a form pertinent for computations. Individual contributions in (3) require regularization which can only be safely removed upon their combination. Hence it is necessary to synchronize the regularization procedure. This is achieved by noting that the Born series for scattering data and the Feynman series are both expansions in powers of the background $V(\mathbf{x})$. Thus we subtract the N leading terms of the Born from the phase shift and add back their contribution as Feynman diagrams:

$$E_{\rm vac} = \frac{1}{2}\sum_{n={\rm b.s.}} |\omega_n| + \int_0^\infty \frac{dk}{2\pi}\,\omega(k)\,\left[\frac{\partial}{\partial k}\delta(k)\right]_N + E_{\rm FD}^{(N)} + E_{\rm ct}\,. \qquad (5)$$

Since the Born series well approximates the phase shifts at large k, a small number N suffices to make the integral finite. The sum $E_{\rm FD}^{(N)} + E_{\rm ct}$ can then be straightforwardly treated by standard renormalization techniques of perturbative quantum field theory. This interesting topic will not central to the present study. But it is important to remember that there is a consistent procedure to render the frequency integrals finite.

2. Electric Potential in Scalar Electrodynamics

The prototype for a frequency interaction with a static background is the electric potential in scalar electrodynamics in one space dimension. In this case the gauge potential is $A_\mu = (V(\mathbf{x}),0)_\mu$ and the local Lagrangian reads

$$\mathcal{L} = (D_\mu\Phi)^*(D^\mu\Phi) - M^2\Phi^*\Phi \qquad \text{with} \qquad D_\mu = \partial_\mu + iA_\mu\,. \qquad (6)$$

The canonical approach to this model is unconventional but straightforward; it can be formulated as an application of the random phase approximation [5]. After Fourier transforming the

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[1] Because of the dispersion relation $\omega = \omega(k)$ we synonymously use ω and k for the frequency.

time–dependence, the field equations turn into wave–equations for bound states $\psi_n(x)$ with discrete energies $|\omega_n| < M$, and for scattering solutions $\psi_k^{(\pm)}(x)$ with energies $\pm\omega_k$ (where $\omega = \omega(k) = \sqrt{k^2 + M^2} > 0$),

$$\left[- \partial_x^2 - V(x)^2 + 2\omega_n V(x) + M^2 \right] \psi_n(x) \;=\; \omega_n^2 \, \psi_n(x) \tag{7}$$

$$\left[- \partial_x^2 - V(x)^2 \pm 2\omega \, V(x) + M^2 \right] \psi_k^{(\pm)}(x) \;=\; \omega^2 \, \psi_k^{(\pm)}(x) \, . \tag{8}$$

Since the theory is not CP-invariant, the existence of a bound state with energy ω_n does not imply the existence of a corresponding bound state with energy $-\omega_n$, and likewise $\psi_k^{(+)}(x) \neq \psi_k^{(-)}(x)$ for the scattering states. Despite of additional time derivatives, canonical momenta are well defined and can be made subject to the conventional equal time commutation relations. It is also possible to construct a Fock space. We refer to Ref.[6] for details of the calculation. Here we just list the result for the vacuum matrix element of the energy density with the continuum contribution expressed in terms of the Green's function

$$u(x) = \sum_n \omega_n \psi_n(x) \big[\omega_n - V(x)\big] \psi_n(x) + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \omega \, \mathsf{Im} \left\{ \big[\omega - V(x)\big] \, G(x,x,\omega) - \omega^2 \, G^{(0)}(x,x,\omega) \right\} . \tag{9}$$

If, as customary, we interpret the imaginary part of the Green's function as the density of states, we immediately recognize that the energy (density) is <u>not</u> given as the single particle energy times the density of states. Recall that this reasoning was fundamental in deriving the phase shift formula from the KFL relation. However, it is obvious that the operation in (4) produces the same coefficient of the Green's function as in (9) since, according to (8) we have to replace $V(\mathbf{x}) \to V(x)^2 - 2\omega V(x)$ before applying the frequency derivative to the wave–equation. As a consequence, the modification in the energy density is exactly compensated by the change in the relation between the derivative of the phase shift and the Green's function. It results in the validity of the phase shift formula for this model.

The interaction Hamiltonian contains time derivatives which spoils the simple relation that the interaction Hamiltonian is the negative interaction Lagrangian. Nevertheless the corresponding relation between the vacuum polarization energy and the quantum action, $S[A_\mu]$

$$E_{\rm vac} = -\frac{1}{T} \left(S[A_\mu] - S[0] \right) , \tag{10}$$

where T is some large time interval, holds.

3. Effective Model from a Heavy Particle

In this section, we consider a model of two scalar boson fields that are coupled via a static but space–dependent background $V(\mathbf{x})$,

$$\mathcal{L} = \frac{1}{2} \, \partial_\mu \Phi \, \partial^\mu \Phi - \frac{M^2}{2} \, \Phi^2 + \frac{1}{2} \, \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \, \varphi^2 + V(\mathbf{x}) \, \Phi \, \varphi \, . \tag{11}$$

We assume that $M \gg m$ for the masses of the scalar fields. Eventually we want to consider an effective theory for φ which inherits a frequency dependent interaction with $V(\mathbf{x})$ from integrating out the heavier field Φ. From the Lagrangian, (11), we obtain the field equations $(\partial^2 + M^2) \, \Phi = V(\mathbf{x}) \, \varphi$ and $(\partial^2 + m^2) \, \varphi = V(\mathbf{x}) \, \Phi$ and the energy density

$$T^{00} = \frac{1}{2} \left(\dot{\Phi}^2 - \Phi \, \ddot{\Phi} + \dot{\varphi}^2 - \varphi \, \ddot{\varphi} \right) + \text{total space derivatives} \, , \tag{12}$$

where we have used the field equations to produce the total derivative terms.

Since the interaction is static we may consider frequency modes $\Phi_\omega(\mathbf{x}) = \int dt\, \Phi(t,\mathbf{x}) \mathrm{e}^{i\omega t}$ and $\varphi_\omega(\mathbf{x}) = \int dt\, \varphi(t,\mathbf{x}) \mathrm{e}^{i\omega t}$. Then the wave–equations become

$$\begin{aligned} -\nabla^2\, \Phi_\omega(\mathbf{x}) &= (\omega^2 - M^2)\, \Phi_\omega(\mathbf{x}) + V(\mathbf{x})\, \varphi_\omega(\mathbf{x})\,, \\ -\nabla^2\, \varphi_\omega(\mathbf{x}) &= (\omega^2 - m^2)\, \varphi_\omega(\mathbf{x}) + V(\mathbf{x})\, \Phi_\omega(\mathbf{x})\,. \end{aligned} \tag{13}$$

Assuming that the heavier field only varies slowly in space defines the local approximation $\Phi_\omega \approx V\, \varphi_\omega/(M^2-\omega^2)$ in which the lighter field is subject to

$$-\nabla^2\, \varphi_\omega \approx (\omega^2 - m^2)\, \varphi_\omega - \frac{V(\mathbf{x})^2}{\omega^2 - M^2 + i\epsilon}\, \varphi_\omega\,. \tag{14}$$

Here we have reinstalled the Feynman pole prescription from the defining functional integral. From this wave–equation we derive an orthogonality relation for modes of different frequencies

$$\int d^dx\, \varphi_\omega(x) \left[1 + \frac{V(\mathbf{x})^2}{(\omega^2 - M^2 + i\epsilon)\,(\omega'^2 - M^2 + i\epsilon)}\right] \varphi_{\omega'}(x) = 0 \qquad \text{for} \qquad \omega \ne \omega'\,. \tag{15}$$

In the same way, we can use the local approximation for Φ_ω directly in the energy density (12)

$$T^{00}(\mathbf{x}) = \int \frac{d\omega}{2\pi}\, T^{00}_\omega(\mathbf{x}) \equiv \int \frac{d\omega}{2\pi}\, \omega^2 \left[1 + \frac{V(\mathbf{x})^2}{(\omega^2 - M^2 + i\epsilon)^2}\right] \varphi_\omega(\mathbf{x})^2\,, \tag{16}$$

where we have omitted contributions that vanish upon spatial integration. Several remarks on this result are in order. First, the straightforward substitution of the frequency decomposition into (12) yields a double frequency integral over ω and ω'. It is only by the orthogonality condition, (15) that the off–diagonal parts, which are explicitly time dependent, vanish. Second, the energy $\int d\mathbf{x}\, T^{00}(\mathbf{x})$ is conserved by the wave–equation (14) in the local approximation and no further input from the full model is required. Third, the kernel of the frequency integral for $T^{00}(\mathbf{x})$ obviously relates to the frequency derivative of the wave–equation (14). It is exactly this derivative that is essential in the derivation, see (4), of the phase shift formula for $\omega^2 = k^2 + m^2$. In total this shows that even in a model in which the frequency dependence is not polynomial and canonical methods do not apply, the phase shift formula for the vacuum polarization energy remains valid. More details on this calculation as well as a numerical comparison of the vacuum polarization energies between the full model and its local approximation in one space dimension can be found in ref.[6].

4. Effective Model for a Dielectric Sphere

A particular interesting topic in the context of vacuum polarization energies is the Casimir self–stress of a dielectric sphere. Some time ago it has been claimed[7] to represent an exception from the rule[8] that Casimir forces are attractive. In this context the quantum fluctuations are photons and the background is a radially function that is strongly peaked at the radius of the sphere. The interaction between the photons and the background occurs via the frequency dependent dielectric function of the material.

The point of departure for this study are Maxwell's equations for a dielectric without sources

$$\nabla\cdot\mathbf{D} = 0\,, \qquad \nabla\times\mathbf{E} + \partial_t\mathbf{B} = 0\,, \qquad \nabla\cdot\mathbf{B} = 0 \quad \text{and} \quad \nabla\times\mathbf{B} - \partial_t\mathbf{D} = 0\,. \tag{17}$$

The dielectric nature enters through the convolution between the displacement and the electric field: $\mathbf{D}(t,\mathbf{x}) = \int dt' \epsilon(t',\mathbf{x})\mathbf{E}(t-t',\mathbf{x})$. Introducing again frequency modes and assuming a spherically symmetric dielectric $\epsilon_k(r) = \int dt\, \mathrm{e}^{ikt}\, \epsilon(t,\mathbf{x})$ leads to the so–called Mie–model[9, 10]. Then Maxwell's equations (17) are transformed into decoupled second order differential equations for scalar fields $\varphi_k(\mathbf{x})$ and $\phi_k(\mathbf{x})$ by the ansätze

$$\text{TE}: \qquad \mathbf{E}_k(\mathbf{x}) = k\,\nabla\times[\varphi_k(\mathbf{x})\,\mathbf{x}] \qquad \text{and} \qquad \mathbf{B}_k(\mathbf{x}) = i\nabla\times(\nabla\times[\varphi_k(\mathbf{x})\,\mathbf{x}])$$

$$\text{TM}: \qquad \mathbf{B}_k(\mathbf{x}) = -ik\,\nabla\times[\phi_k(\mathbf{x})\,\mathbf{x}] \qquad \text{and} \qquad \mathbf{E}_k(\mathbf{x}) = \frac{-1}{\epsilon_k(r)}\nabla\times(\nabla\times[\phi_k(\mathbf{x})\,\mathbf{x}])\,, \tag{18}$$

for the transverse electric (TE) and transverse magnetic (TM) modes, respectively. The scalar functions obey simple differential equations that can be cast into

$$\left(k^2+\nabla^2\right)\varphi_k(\mathbf{x}) = U_k^{(TE)}\varphi_k(\mathbf{x}) \qquad \text{and} \qquad \left(k^2+\nabla^2\right)\phi_k(\mathbf{x}) = U_k^{(TM)}\phi_k(\mathbf{x})\,. \tag{19}$$

The explicit expressions for the potential in terms of the dielectric function are given in ref.[11].

Assuming that the dielectric function has a power expansion in k^2, which in coordinates space is $-\partial_t^2$, it is possible to identify a conserved energy from Maxwell's equations (17). A single frequency mode contributes the spatial integral of

$$u_k(\mathbf{x}) = \frac{1}{2}\frac{\partial[k\epsilon_k(r)]}{\partial k}\,\mathbf{E}_k^2 + \frac{1}{2}\mathbf{B}_k^2 \tag{20}$$

to the spatial density of that energy. Expressing this energy density in terms of the scalar functions ϕ_k and φ_k and using the respective wave–equations to eliminate surface terms (in coordinate space) shows that the metric functions M_k in $u_k^{(TE)} = \frac{1}{2}M_k^{(TE)}\phi_k^2$ and $u_k^{(TM)} = \frac{1}{2}M_k^{(TM)}\varphi_k^2$ are indeed the frequency derivatives $M_k = \frac{k}{2}\frac{\partial}{\partial k}\left(k^2 - U_k\right)$ in both the TE and TM channels. These are the frequency derivatives of the wave–equations (19)[6]. Hence the phase shift formula does not get modified for this effective model.

However there are problems with this effective model. The assumption that ϵ_k is even in k is unphysical as it violates the Kramers–Kronig relations. The analog construction that allows for odd contributions requires ϵ_k to be a tensor with anti–symmetric components. However, by general arguments based on the Onsager relations in statistical mechanics, ϵ_k is established to be a symmetric tensor[12]. In that context it is more appropriate to study the free energy[13]. Then one considers the analytic continuation ($k = i\kappa$) of the standard energy density

$$\overline{u}_{i\kappa}(\mathbf{x}) = \frac{1}{2}\epsilon_{i\kappa}(r)\,\mathbf{E}_{i\kappa}^2 + \frac{1}{2}\mathbf{B}_{i\kappa}^2 \tag{21}$$

as the contribution of a single mode to the vacuum polarization energy. Formally it adds

$$\Delta u = \int_0^\infty \frac{d\kappa}{\pi}\sum_{\ell=1}^{\infty}(2\ell+1)\kappa^2\int_0^\infty dr\left(\epsilon_{i\kappa}(r) - 1 - \frac{1}{2\kappa}\frac{dU_{i\kappa}(r)}{d\kappa}\right)|\psi_{i\kappa,\ell}(r)|^2 \tag{22}$$

to the unrenormalized energy. Note that Δu implies the sum over TE and TM modes and the $\psi_{i\kappa,\ell}$ are the analytic continuations of the radial parts of the angular momentum (ℓ) decompositions of ϕ_k and φ_k. The renormalization and the numerical computations have been performed in ref.[11]. The result is shown in figure 1.

While the naive application of the phase shift formula, *i.e.* adopting the energy functional from (20), indeed produces self–repulsion for the dielectric sphere as suggested some time ago[7] the additional term form (22) overcomes this contribution predicting an attraction in total.

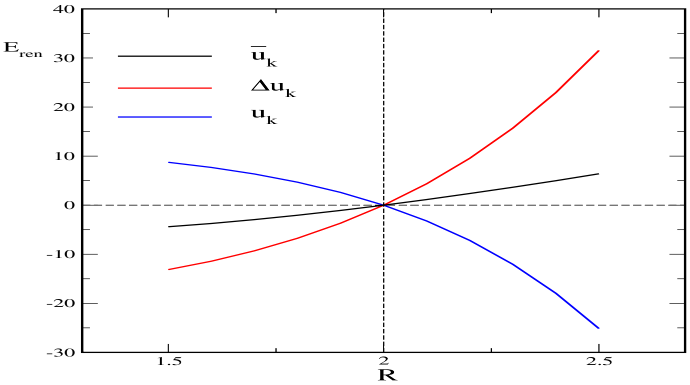

Figure 1. Renormalized vacuum energy of the dielectric sphere of radius R. The legend associates the kernels for the energy integrals: u_k from (20), Δu from (22) and $\overline{u}$ from (21) is the total energy. Units are such that the plasma frequency of the dielectric is $\lambda = 2$.

5. Conclusion

In a rigorous field theory approach the phase shift formula for the vacuum polarization energy remains valid even when the interaction is frequency dependent. Here we have only considered a few examples. However, once the wave–equations produce a conserved energy functional (which may not always be the case) its kernel in frequency space is related to the frequency derivative of the wave–equations for the following consistency condition: This derivative features in the orthogonality condition for the solutions to the wave–equation and in this way ensures that the terms which are explicitly time dependent disappear from the total energy. On the other hand it is exactly this derivative that enters the derivation of the phase shift formula via the Wronskian between the regular and Jost solutions. Hence any additional term in that Wronskian identically appears in the energy functional thereby maintaining the phase shift formula for the vacuum polarization energy.

In particular cases effective theories require input from outside local quantum field theory. Then the phase shift formula is indeed modified. For the particular case of a dielectric shell this modification turns the self–stress from being repulsive to attractive.

Acknowledgments

This presentation is based on an ongoing collaboration with N. Graham and M. Quandt. Their input is highly appreciated. Support from the National Research Foundation (NRF), under Ref. No. IFR1202170025 is acknowledged.

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