Homogeneous Open Quantum Walks on a Line

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Abstract. We analyze the dynamics of homogeneous open quantum walks on a line for a system with two internal degrees of freedom by analytically computing the probability distribution of the system. These distributions are plotted numerically, allowing for the interpretation of the dynamics of the system at any timestep. We also compute for the system's steady point at each point of the line on which the open quantum walk takes place.

1. Introduction

Recently, the open quantum walk was introduced in [1, 2], wherein the system undergoes a random walk, and its internal state changes with each step. The dynamics is driven by the coupling between the system and its associated environment. This open quantum walk may occur over discrete or continuous time steps in a finite or infinite graph, with the continuous-time limit investigated in [3, 4]. Previous work on open quantum walks resulted in the derivation of a central limit theorem for its asymptotic probability distribution, which is shown to be a normal distribution [5, 6]. Aside from showing rich dynamical behavior, possible applications for open quantum walks have been discussed in [1]. Those possible applications are dissipative quantum state preparation of single-and multiple-qubit gates, implementation of quantum logic gates for single and multiple qubits, and efficient quantum transport of excitations.

A particular type of open quantum walk was studied in [7]. The open quantum walk under consideration was a homogeneous open quantum walk on a line, with one jump operator corresponding to one direction of motion for the system. The jump operators were assumed to be simultaneously diagonalizable. The resulting distribution was shown to be, for intermediate timesteps, a binomial distribution, which converges for large timesteps to a Gaussian, distribution. The approach then allows us to determine the dynamics of the system undergoing the open quantum walk at any instant of time.

In this paper, we extend the work done in [7] to consider systems with 2 internal degrees of freedom undergoing homogeneous open quantum walks on the line with at least one of the jump operators diagonalizable. We analyze the dynamics of the system undergoing the open quantum walk, and we also derive the steady state of the system at each point of the line on which this open quantum walk takes place.

2. The Homogeneous Open Quantum Walk

A schematic diagram of the homogeneous open quantum walk is shown in figure (1). We designate the Hilbert space corresponding to the position of the system undergoing the open quantum walk as $\mathcal{H}_s = \mathbb{Z}$, which we designate as position space, and the Hilbert space corresponding to the internal degrees of freedom of the system at each node as $\mathcal{H}_c$, which we designate as node space. The total Hilbert space corresponding to the open quantum walk is then $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_s$. If the open quantum walk occurs over discrete timesteps, then at timestep n, the density matrix describing the system undergoing this open quantum walk is given as

$$\rho^{(n)} = \sum_{x=-M}^{M} \rho_x \otimes |x\rangle \langle x| \,, \tag{1}$$

where ρ_x is the density matrix that describes the system's internal degrees of freedom at node x, $|x\rangle$ is the ket in position space $\mathcal{H}_s$, and $2M+1$ is the total number of nodes on which the system can move at timestep n. We note that the density matrices ρ_x satisfy the condition $\sum_{x=-M}^{M} \mathrm{Tr}(\rho_x) = 1$.

In $\mathcal{H}_c$, we define two bounded operators B and C that satisfy the condition

$$B^\dagger B + C^\dagger C = I, \tag{2}$$

with this condition ensuring that probability is conserved at all timesteps n. These operators B and C correspond to the change in the system's internal degrees of freedom as it makes a transition from node x to a neighboring node $x \pm 1$. For these jump operators, we define a linear mapping $\mathcal{L}$ on $\mathcal{H}_c$ as

$$\mathcal{L}(\rho) = B\rho B^\dagger + C\rho C^\dagger \tag{3}$$

Lifting this mapping from $\mathcal{H}_c$ to $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$, and iteratively applying the resulting map, we obtain an equation of motion for the system undergoing the open quantum walk at timestep n and node x. In particular, the jump operators in the system's Hilbert space $\mathcal{H}$ will now have the form $B \otimes |x+1\rangle \langle x|$ and $C \otimes |x-1\rangle \langle x|$, while the mapping now has the form

$$\mathcal{M}(\rho^{(n-1)}) = \sum_{x=-n}^{n} \rho_x^{(n)} \otimes |x\rangle \langle x| \,, \tag{4}$$

where

$$\rho_x^{(n)} = B\rho_{x-1}^{(n-1)} B^\dagger + C\rho_{x+1}^{(n-1)} C^\dagger, \tag{5}$$

which is the time evolution equation for the density matrix at node x at timestep n. Also,

$$P_x^{[n]} = \mathrm{Tr}(\rho_x^{[n]}) \tag{6}$$

is the probability that node x is occupied at timestep n.

If B is diagonalizable via a unitary transformation U, (2) can be transformed into the following form:

$$\tilde{B}^\dagger \tilde{B} + \tilde{C}^\dagger \tilde{C} = I, \tag{7}$$

where $\tilde{B} = U^\dagger B U$ and $\tilde{C} = U^\dagger C U$. In particular, $\tilde{B}$ has the form

$$\tilde{B} = b_1 |1\rangle \langle 1| + b_2 |2\rangle \langle 2| \,, \tag{8}$$

where b_j are the eigenvalues of B. We note that the states $|1\rangle$ and $|2\rangle$ are internal states of the system, which are 2×1 unit vectors in the node Hilbert space $\mathcal{H}_c$. On the other hand, $\tilde{C}$ will have the following form:

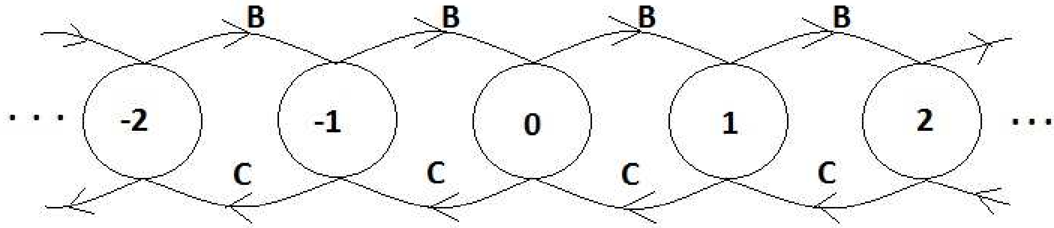

Figure 1. A homogeneous open quantum walk on a line.

$$\tilde{C} = \sqrt{1-|b_1|^2}(\alpha |1\rangle \langle 1| - \beta^* |2\rangle \langle 1|) + \sqrt{1-|b_2|^2}(\beta |1\rangle \langle 2| + \alpha^* |2\rangle \langle 2|), \quad (9)$$

where α, β are complex numbers that obey the condition $|\alpha|^2 + |\beta|^2 = 1$. In general, the jump operators for a homogeneous open quantum walk do not commute, except for the case where both B and C are simultaneously diagonalizable, which has been covered in Ref. [7]. It is still possible, as we demonstrate for a particular case below, the probability distributions can be computed exactly for arbitrary timesteps n.

3. Computation of probability distributions

In this section we present two methods for computing the probability distributions for systems undergoing open quantum walks. The first method, which uses the central limit theorem, allows us to determine the asymptotic probability distribution over time for the system. The second method, which makes use of Fourier transforms, gives us a brute force method for computing the system's probability distribution.

3.1. Central Limit Theorem for Jump Operators

Ref. [5] gives us a central limit theorem that tells us the asymptotic behavior of the probability distribution for a system underoing an open quantum walk. In particular, for a homogeneous open quantum walk, the theorem can be stated as follows:

Theorem *(Attal et al, Ref. [5], Theorem 5.2). Consider the stationary open quantum random walk on Z associated to the jump operators B and C. We assume that the completely positive map $\mathcal{L}(\rho) = B\rho B^\dagger + C\rho C^\dagger$ admits a unique invariant state ρ_∞. Let (ρ_n, X_n) be the quantum trajectory process to this open quantum walk. Then $\frac{X_n - nm}{\sqrt{n}}$ converges in law to the Gaussian distribution $\mathcal{N}(0, C)$ in $\Re$, with mean $m = \mathrm{Tr}(B\rho_\infty B^\dagger) - \mathrm{Tr}(C\rho_\infty C^\dagger)$ and covariance $\sigma^2 = \mathrm{Tr}(B\rho_\infty B^\dagger + C\rho_\infty C^\dagger) - m^2 + 2\mathrm{Tr}(B\rho_\infty B^\dagger L - C\rho_\infty C^\dagger L) - 2m\mathrm{Tr}(\rho_\infty L)$ where L is the solution to the equation $L - \mathcal{L}^\dagger(L) = B^\dagger B - C^\dagger C - I$.*

In the theorem, (ρ_n, X_n) is the Markov chain with values on $\mathcal{E}(\mathcal{H}_c) \times \mathbb{Z}$, where $\mathcal{E}(\mathcal{H}_c)$ is the space of all density matrices on $\mathcal{H}_c$, associated with the quantum trajectories of the mapping $\mathcal{M}$.

For intermediate timesteps, on the other hand, we can determine the dynamical behavior of the system undergoing the open quantum walk by analytically computing for it. Also, using the mapping given by (3), we can compute for the form of the steady state for the open quantum walk at each node of the line by solving the system of equations specified by the following equation:

$$\rho_\infty = \mathcal{L}(\rho_\infty) = B\rho_\infty B^\dagger + C\rho_\infty C^\dagger. \quad (10)$$

3.2. Fourier Transform Method for Computing Probability Distributions
To compute the probability distributions for the open quantum walk generated by the jump operators B and C, we use the method of Fourier transforms and dual processes first described in Ref. [6]. First, we compute for the following dual process:

$$Y_n(k) = (e^{ik} L_{B^\dagger} R_B + e^{-ik} L_{C^\dagger} R_C)^n (I), \tag{11}$$

where for a given operator A acting on U, $L_A U := AU$ and $R_A U := UA$. In (11), I is the $N \times N$ identity matrix. This dual is defined in momentum space K, where $k = \{-\pi, \pi\}$. Once we have determined the dual $Y_n(k)$, we then compute for the trace of the product $\rho_0 Y_n(k)$, where ρ_0 is the density matrix for the initial state of the system, with explicit form

$$\rho_0 = P_1|1\rangle\langle 1| + P_2|2\rangle\langle 2| + q_{12}|1\rangle\langle 2| + q_{12}^*|2\rangle\langle 1|. \tag{12}$$

The trace of ρ_0 is then $P_1 + P_2 = 1$. Finally, we compute for the probability distribution of the system in position space at point x and timestep n by taking the Fourier transform of $\mathrm{Tr}(\rho_0 Y_n(k))$ as follows:

$$P_x^{(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \;\; e^{-ikx} \mathrm{Tr}(\rho_0 Y_n(k)). \tag{13}$$

3.3. Computation of the probability distribution for a particular form of jump operators
To demonstrate how we can compute for the probability distribution of a system undergoing an open quantum walk analytically, let us consider transformed jump operators of the form

$$\tilde{B} = b_1(|1\rangle\langle 1| - |2\rangle\langle 2|), \;\; \tilde{C} = \sqrt{1 - |b_1|^2}(|1\rangle\langle 2| + |2\rangle\langle 1|). \tag{14}$$

With this form of B and C, it can easily be seen that B and C both satisfy (2), and B and C will not commute. Let us now compute for the probability distribution $P_x^{(n)}$. Evaluating (11), then substituting the resulting dual form and the initial state given by (12) in (13), we obtain the following probability densities at timestep n and position x:

$$P_x^{(n)} = \binom{n}{(n-x)/2} (|b_1|^2)^{(n-x)/2} (1 - |b_1|^2)^{(n+x)/2}. \tag{15}$$

This is a binomial distribution, which converges to a normal distribution as $n \to \infty$. Hence, for this very special case, we have demonstrated that it is possible to compute for the analytic form of the probability distribution for a system undergoing a homogeneous open quantum walk at any instant of time.

4. Numerical Analysis of the Probability Distributions of a Homogeneous Open Quantum Walk

In general, even if B and C have been transformed into the forms given by Eqs. (8) and (9), the resulting analytic form of the probability distribution is not as simple as that given in (15). In fact, the resulting expression only reduces to well known simple probability distributions at any instant of time n only for certain special cases. Thus, to properly analyze the probability distribution, we must plot it at a given instant of time n and see how the distribution evolves numerically, in order to obtain a physical interpretation for the system's dynamical behavior.

The first important feature to note is that if neither b_1 nor b_2 are equal to either zero or 1, the distribution converges numerically to a normal distribution with a well-defined peak for large timesteps. This is not surprising in light of Attal et al's central limit theorem for open quantum

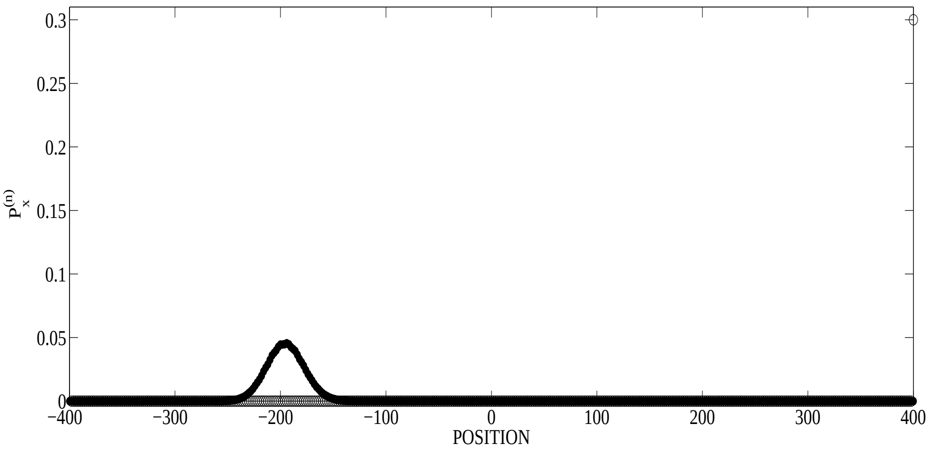

Figure 2. Probability distributions for a homogeneous open quantum walk at timestep $n = 400$, with B and C given by Eqs. (8) and (9), respectively. The solid circles denote a distribution corresponding to $\tilde{B}$ with eigenvalues $b_1 = 0.45$ and $b_2 = 0.55$, while the unshaded circles denote a solitonic distribution corresponding to $\tilde{B}$ with eigenvalues $b_1 = 1$ and $b_2 = 0$. For both distributions, the initial state $\rho_0^{(0)}$'s matrix elements are $\rho_{11} = 0.3$, $\rho_{22} = 0.7$, $\rho_{12} = 0.33 + 0.11i$ and $\rho_{21} = \rho_{12}^*$.

walks. However, by plotting the analytic form of the distribution for any given timestep, we can see how the distribution will evolve towards a normal distribution.

Second, we note that if $b_1 = 1$ and $b_2 = 0$, or vice versa, in (8), then $\alpha = 1$ and $\beta = 0$, or vice versa, in (9). We then obtain a solitonic distribution, which is an infinitely narrow peaked distribution with constant height moving with constant speed to the left or to the right. We illustrate this in Fig. (2), where we plot a normal distribution and a solitonic distribution, the latter represented by the single point on the right hand side.

5. Steady States for a Homogeneous Open Quantum Walk in $\mathcal{H}_c$ space

We now turn to the question of determining the steady states ρ_∞ for the linear mapping $\mathcal{L}$ in $\mathcal{H}_c$ space of this homogeneous open quantum walk, and in so doing determine whether the central limit theorem for open quantum walks holds for this case. To do so, we must solve (10), and to simplify our task, we transform (10) by diagonalizing B, such that B and C will be transformed into the forms $\tilde{B}$ and $\tilde{C}$ given by Eqs. (8) and (9) for the diagonal blocks of $\tilde{C}$, respectively. Then (10) will have the form

$$\tilde{\rho}_\infty = \tilde{B}\tilde{\rho}_\infty\tilde{B}^\dagger + \tilde{C}\tilde{\rho}_\infty\tilde{C}^\dagger, \quad \tilde{\rho}_\infty = U\rho_\infty U^\dagger. \tag{16}$$

We assume that $\tilde{\rho}_\infty$ has the general form

$$\tilde{\rho}_\infty = \rho_{11} |1\rangle \langle 1| + \rho_{12} |1\rangle \langle 2| + \rho_{21} |2\rangle \langle 1| + \rho_{22} |2\rangle \langle 2| . \tag{17}$$

Substituting Eqs. (8), (9) and (17) into (16) and simplifying, we obtain a system of 4 equations in 4 unknowns for the matrix elements ρ_{jk} of $\tilde{\rho}_\infty$:

$$\begin{aligned}
&0 = (1-|b_1|^2)(|\alpha|^2-1)\tilde{\rho}_{11} + (1-|b_2|^2)|\beta|^2\tilde{\rho}_{22} + \alpha\beta^*\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{12} \\
&+ \alpha^*\beta\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{21} \\
&0 = |\beta|^2(1-|b_1|^2)\tilde{\rho}_{11} + (1-|b_2|^2)(|\alpha|^2-1)\tilde{\rho}_{22} - \alpha\beta^*\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{12} \\
&- \alpha^*\beta\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{21} \\
&0 = -\alpha\beta(1-|b_1|^2)\tilde{\rho}_{11} + \alpha\beta(1-|b_2|^2)\tilde{\rho}_{22} + \left[b_1b_2^* - 1 + \alpha^2\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\right]\tilde{\rho}_{12} \\
&- \beta^2\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{21} \\
&0 = -\alpha^*\beta^*(1-|b_1|^2)\tilde{\rho}_{11} + \alpha^*\beta^*(1-|b_2|^2)\tilde{\rho}_{22} - (\beta^*)^2\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\tilde{\rho}_{12} \\
&+ \left[b_1^*b_2 - 1 + (\alpha^*)^2\sqrt{(1-|b_1|^2)(1-|b_2|^2)}\right]\tilde{\rho}_{21}
\end{aligned} \tag{18}$$

However, the second of these four equations is actually linearly dependent on the first, so that at this point, we have 3 equations in 4 unknowns. In order to return the system to 4 equations in 4 unknowns, we impose the following additional constraint, following Ref. ([1]):

$$\mathrm{Tr}(\tilde{\rho}_\infty) = \rho_{11} + \rho_{22} = 1 \tag{19}$$

Thus, solving the resulting system of linear equations gives us a unique solution for $\tilde{\rho}_\infty$:

$$\tilde{\rho}_\infty = \frac{(1-|b_1|^2)(1-|b_2|^2)}{2-|b_1|^2-|b_2|^2}\left(\frac{1}{1-|b_1|^2}\left|1\right\rangle\left\langle 1\right| + \frac{1}{1-|b_2|^2}\left|2\right\rangle\left\langle 2\right|\right). \tag{20}$$

6. Conclusion

We have shown in this work that, for a particular homogeneous open quantum walk for a system with 2 internal degrees of freedom, with one of the jump operators diagonalizable, it is possible to compute analytically for the probability distribution of the open quantum walk for a given timestep n. At the same time, we have also shown numerically that such a probability distribution will converge for large timesteps to a normal distribution with 1 peak or a solitonic distribution if the non-diagonalizable jump operator for this open quantum walk is transformable to a particular form given by (9). We also determined a unique steady state of the mapping $\mathcal{L}$ at each node of the line for this homogeneous open quantum walk.

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