

Quantum measurements along accelerated world-lines

JM Hartman, SH Connell and F Petruccione

University of Johannesburg, Johannesburg, South Africa

E-mail: hartman.jonathan2@gmail.com

Abstract. In this research, we are working with a formalism for quantum measurements that takes special relativity into account. The ultimate goal is to modify this framework to work with more general space-times rather than just Minkowski space-time and determine how the metric would affect quantum entanglement by doing calculation of Bell's Theorem in curved space-time. As a first step in that direction, in this paper, we calculate the case for quantum measurements along an accelerated world line by solving the Schwinger-Tomonaga equation.

1. Introduction

Most of modern physics can be described either within the framework of general relativity or within the framework of quantum mechanics. General relativity describes one of the four fundamental forces, gravitation, as a warpage of space-time whereas the other three, namely the electromagnetic, and the strong and weak nuclear forces can be adequately described within the framework of quantum mechanics. However, combining general relativity with quantum mechanics in order to formulate a theory of quantum gravity has proven difficult. The ultimate goal of the current work is to modify a framework of relativistic quantum mechanics, though not full quantum field theory, such that it includes metrics other than the Minkowski metric as the space-time background. This relativistic framework was formulated by Breuer and Petruccione [1] [2]. The goal is also to determine what the effect of the space-time background, if any, is on the measurements of entangled particles.

This paper summarises the framework, both non-relativistic and relativistic. In the non-relativistic case, the framework is a statistical multi-particle formalism of quantum mechanics which is formulated in terms of probabilities and allows for interactions between different quantum particles. The ultimate goal of this research is to extend this special relativistic framework to work with curved space-time backgrounds and then find out what, if any, effect the metric itself has on the phenomenon of quantum entanglement. That means to work with more generalised space-time backgrounds rather than just Minkowski space-time.

2. Background

The non-relativistic framework is mainly governed by the Schrödinger equation,

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \quad (1)$$

there are three different formulations, namely the Schrödinger picture, the Heisenberg picture and the Interaction picture. In the Schrödinger picture, the state vectors evolve in time, while

the operators remain consistent with respect to time. The solution to the Schrödinger equation may be expressed in terms of the unitary time-evolution operator $U(t, t_0)$ that transforms the state vector $|\psi(t_0)\rangle$ at some initial time t_0 to the state $|\psi(t)\rangle$ at time t :

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle. \quad (2)$$

If we substitute equation (2) into the Schrödinger equation (1), we get the differential equation

$$i \frac{\partial}{\partial t} U(t, t_0) = H(t) U(t, t_0), \quad (3)$$

where we take the initial condition as $U(t_0, t_0) = I$ with I as the identity. For a closed isolated system, the differential equation (3) can be integrated to yield the solution

$$U(t, t_0) = \exp[-iH(t - t_0)]. \quad (4)$$

For a time-dependent Hamiltonian, the solution to equation (3) subject to the initial condition above, is

$$U(t, t_0) = T_{\leftarrow} \exp \left[-i \int_{t_0}^t ds H(s) \right], \quad (5)$$

where T_{\leftarrow} is the chronological time-ordering operator that orders the products of time dependent operators from right to left, in the direction of the arrow. If the system is in a mixed state, then the quantum statistical ensemble may be characterised with the help of the statistical operator ρ , which is the generalisation of the density operator to include continuous spectra.

On the other hand, the operators in the Heisenberg picture evolve in time while the state vectors remain constant with respect to time. The Schrödinger and Heisenberg pictures are related to each other and the Schrödinger picture can be converted to the Heisenberg picture by transferring the time dependence of the state vectors to the operators. So, if $A(t)$ is an observable represented in the Schrödinger picture, then it can be converted to the Heisenberg picture by,

$$A_H(t) = U^\dagger(t, t_0) A(t) U(t, t_0) \quad (6)$$

where $U(t, t_0) = T_{\leftarrow} \exp \left[-i \int_{t_0}^t ds H(s) \right]$ is the time evolution operator with T_{\leftarrow} as the chronological time ordering operator.

A given Schrödinger operator $A(t)$ can be transformed into the Heisenberg equivalent $A_H(t)$ by the relation (6). The two pictures are assumed to coincide at initial time t_0 , such that $A_H(t_0) = A(t_0)$. The Heisenberg equation of motion can be derived straightforwardly from the transformation law (6) by differentiating both sides of the equation, from which we obtain

$$\frac{d}{dt} A_H(t) = i [H_H(t), A_H(t)] + \frac{\partial A_H(t)}{\partial t}, \quad (7)$$

where $H_H(t) = U^\dagger(t, t_0) H(t) U(t, t_0)$ is the Hamiltonian in the Heisenberg picture, $\frac{d}{dt}$ is the total time derivative and $\frac{\partial}{\partial t}$ is the partial time derivative with respect to the explicit time dependence of the Schrödinger operator.

The Schrödinger and Heisenberg pictures are limiting cases of a more general formulation called the Interaction picture, where both state vectors and operators can evolve in time. The Interaction picture is more useful in this context because it can more easily deal with the changes to wave functions and observables due to interactions. Thus, with the Interaction picture one can construct solutions to the many-body Schrödinger equation, usually as the solution to the free

particle problem plus an interaction term. The Hamiltonian can be given by $H(t) = H_0 + \hat{H}_I(t)$, where $\hat{H}_I(t)$ is Hamiltonian of the interaction between the systems. Then, if we introduce the unitary time evolution operators as $U_0(t, t_0) \equiv \exp[-iH_0(t - t_0)]$ and $U_I(t, t_0) \equiv U_0^\dagger U(t, t_0)$, then we convert an operator $A(t)$ in the Schrödinger picture representation to the equivalent form in the Interaction picture by the relation:

$$A_I \equiv U_0^\dagger(t, t_0)A(t)U_0(t, t_0) \quad (8)$$

The Interaction picture time evolution operator is the solution to the differential equation

$$i\frac{\partial}{\partial t}U_I(t, t_0) = H_I(t)U_I(t, t_0) \quad (9)$$

and the Interaction picture Hamiltonian is given by

$$H_I(t) \equiv U_0^\dagger(t, t_0)\hat{H}_I(t)U_0(t, t_0). \quad (10)$$

In the relativistic case, we work with a framework where the quantum system is represented as a functional on the set of space like hypersurfaces in Minkowski space. In the relativistic framework we can denote the coordinates of a point x in Minkowski space-time as the four-vector $x^\mu = (x^0, \hat{x})$, where $x^0 = t$ is the time coordinate and $\vec{x} = (x^1, x^2, x^3)$ are the space coordinates. We use the units such that $\hbar = c = 1$ where \hbar is Planck's constant and c is the speed of light. The Lorentz invariant inner product of two four-vectors x and y is given by

$$xy = x^\mu y_\mu = \eta_{\mu\nu}x^\mu y^\nu = x^0 y^0 - \vec{x} \cdot \vec{y} \quad (11)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. The state vector $|\Psi(t)\rangle$ evolves in the Interaction picture according to the Schrödinger equation

$$\frac{\partial}{\partial t}|\Psi(t)\rangle = -iH_I|\Psi(t)\rangle, \quad (12)$$

where the Interaction Hamiltonian is given by

$$H_I = \int d^3x \mathcal{H}(t, \vec{x}) \quad (13)$$

with $\mathcal{H}(x) = \mathcal{H}(t, \vec{x})$ as the Hamiltonian density of the theory. The state vector $|\Psi(t)\rangle$ gives the state of a quantum system at a fixed time $x^0 = t$ and so allows the evaluation of expectation values of observables on the three dimensional hyper surface at constant t . The Lorentz invariant generalisation of this concept is that of a state vector associated with a general three dimensional space-like hypersurface σ which is also a manifold in Minkowski space. Therefore, the state vector becomes a functional $|\Psi\rangle = |\Psi(\sigma)\rangle$ on the space of all such hypersurfaces. The Schrödinger equation (12) then becomes a functional differential equation, called the Schwinger-Tomonaga equation (50) [3] [4] [5] [6].

3. Statistical formulation of non-relativistic quantum mechanics

In quantum mechanics, the state of a closed physical system is described by the wave function ψ , also given as the state vector, represented by the ket $|\psi\rangle$, which is an element of a Hilbert space \mathbf{H} . The observables are represented as linear self-adjoint operators in the Hilbert space. An important theorem regarding self-adjoint operators is the spectral theorem, which is a generalisation of the eigenvalue theorem in linear algebra to operators in general. The spectral theorem states that for any self-adjoint operator \hat{R} , there is a spectral family E_r such that

$$\hat{R} = \int_{-\infty}^{\infty} r dE_r. \quad (14)$$

The spectral family E_r , where $r \in \mathbb{R}$, is a one-parameter family of commuting orthogonal projection operators such that they are monotonically increasing, $E_{r'} \geq E_r$ for $r' > r$, continuous from the right, $\lim_{\epsilon \rightarrow +0} E_{r+\epsilon} = E_r$, and has the limits $\lim_{r \rightarrow -\infty} E_r = 0$ and $\lim_{r \rightarrow \infty} E_r = I$, where I is the identity operator.

The spectral decomposition (14) is important for the statistical interpretation of quantum mechanics. For the statistical interpretation of quantum mechanics, consider a statistical ensemble $\mathcal{E} = \{S^{(1)}, S^{(2)}, \dots, S^{(N)}\}$, where $S^{(1)}, S^{(2)}, \dots, S^{(N)}$ are identically prepared quantum systems. Given the ensemble \mathcal{E} , the statistical interpretation of quantum mechanics is based on two postulates. The first postulate is that under certain conditions, a complete characterisation of the statistical ensemble is given by a normalised state vector $|\psi\rangle$ in the Hilbert space \mathbf{H} of the quantum mechanical system. The second postulate is that the measurable quantities of the statistical ensemble \mathcal{E} are represented by self-adjoint operators in the Hilbert space \mathbf{H} . So, if we make measurements of an observable \hat{R} , performed on the ensemble characterised by $|\psi\rangle$, then the outcomes of those measurements represent a real-valued random variable R with a cumulative distribution function $F_R(r)$, given by

$$F_R(r) = \langle \psi | E_r | \psi \rangle \quad (15)$$

where E_r is the spectral family of \hat{R} . By means of the distribution function (15), one can define the probabilities for more general events. If B denotes a Borel set of \mathbb{R} , then we can define a projector operator as

$$E(B) = \int_B dE_r, \quad (16)$$

such that the probability of the measurement outcome falling into the set B is given by

$$P_R(B) = \langle \psi | E(B) | \psi \rangle, \quad (17)$$

where E_r is, again, the spectral family of the operator \hat{R} . Here, the projection of the union of any sequence of disjoint Borel sets B_i is equal to the sum of the individual projections,

$$E\left(\bigcup_i B_i\right) = \sum_i E(B_i). \quad (18)$$

Therefore, if we apply a equation (17) to the relation (18), we find that a similar relation holds for the probability of the union of self-adjoint Borel sets,

$$P_R\left(\bigcup_i B_i\right) = \left\langle \psi \left| \sum_i E(B_i) \right| \psi \right\rangle = \sum_i \langle \psi | E(B_i) | \psi \rangle = \sum_i P_R(B_i). \quad (19)$$

So, the random variable R that results from the the outcomes of measurements on the observable \hat{R} with the distribution function (15) also describes the probabilities for all possible measurement outcomes. Using the spectral decomposition, the mean value of R is given by

$$E(R) = \int_{-\infty}^{\infty} r dF_R(r) = \int_{-\infty}^{\infty} r \langle \psi | E_r | \psi \rangle = \langle \psi | \hat{R} | \psi \rangle. \quad (20)$$

Therefore, the variance of R is given by

$$\text{Var}(R) = E(R^2) - [E(R)]^2 = \langle \psi | \hat{R}^2 | \psi \rangle - \langle \psi | \hat{R} | \psi \rangle^2. \quad (21)$$

The above analysis of quantum statistical ensembles only applies to pure states. In order to obtain more general ensembles, including those of mixed states, consider a number M of ensembles $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_M$, all of the same type as described above and each being described by a normalised state vector $\psi_\alpha, \alpha = 1, 2, \dots, M$ in the Hilbert space \mathbf{H} . The statistics of the total ensemble \mathcal{E} is then obtained by mixing all the constituent ensembles \mathcal{E}_α with respect to their statistical weights w_α which satisfy the condition

$$w_\alpha \geq 0, \sum_{\alpha=1}^M w_\alpha = 1. \quad (22)$$

We mix the ensembles by taking a large number N_α of systems from each of the ensembles \mathcal{E}_α . The total number of systems, $N = \sum_\alpha N_\alpha$, then makes up the new ensemble \mathcal{E} with the statistical weights given by $w_\alpha = \frac{N_\alpha}{N}$. Measurements of an observable \hat{R} on the ensemble \mathcal{E} now produces the random variable R with the distribution function

$$F_R(r) = \sum_{\alpha} w_\alpha \langle \psi_\alpha | E_r | \psi_\alpha \rangle. \quad (23)$$

According to equation (20), the mean is then given by

$$E(R) = \sum_{\alpha} w_\alpha \langle \psi_\alpha | \hat{R} | \psi_\alpha \rangle. \quad (24)$$

These formulae can be written more concisely if we introduce the quantum mechanical analogue of the phase-space probability measure in classical statistical mechanics, called the density matrix. The density matrix is defined as

$$\rho = \sum_{\alpha} w_\alpha |\psi_\alpha\rangle \langle \psi_\alpha|. \quad (25)$$

In terms of the density matrix, the distribution function (23) can now be re-written as

$$F_R(r) = \text{tr}\{\hat{R}\rho\}, \quad (26)$$

where the trace of an operator A is defined as $\text{tr}(A) = \sum_i \langle \varphi_i | A | \varphi_i \rangle$ and $\{\varphi_i\}$ as an orthonormal basis of the Hilbert space \mathbf{H} . Likewise, the analogue of the mean and variance of the random variable R , equations (20) and (21) respectively, in the new ensemble \mathcal{E} are given by

$$E(R) = \text{tr}\{\hat{R}\rho\} \quad (27)$$

and

$$\text{Var}(R) = \text{tr}\{\hat{R}^2\rho\} - \left[\text{tr}\{\hat{R}\rho\} \right]^2 \quad (28)$$

respectively.

4. Quantum Measurements

In an ideal measurement in quantum mechanics, if a property B with corresponding projection operator $E(B)$ is measured on a quantum statistical ensemble \mathcal{E} described by density matrix ρ , then after the measurement, we find that the density matrix ρ' which describes the ensemble \mathcal{E}' that consists of the systems for which the property B is found to be true is given by

$$\rho' = \frac{E(B)\rho E(B)}{\text{tr}\{E(B)\rho E(B)\}}, \quad (29)$$

where the projection operator $E(B)$ is defined in terms of the spectral family of \hat{R} by equation (17). Equation (29) is called the von Neumann-Lüders projection postulate [7] [8]. The above describes the ideal measurement of the projection $E(B)$ derived from the spectral family of \hat{R} but in practice one can only measure an approximation that involves the finite resolution of the detector. If we consider a measurement scheme which yields a set \mathcal{M} of possible outcomes $m \in \mathcal{M}$, then the von Neumann-Lüders projection postulate (29) can be generalised as follows:

- (i) The measurement outcome m represents a classical random number with probability distribution

$$P(m) = \text{tr}\{F_m\rho\}, \quad (30)$$

where F_m is a positive operator, called the effect, which satisfies the normalisation condition

$$\sum_{m \in \mathcal{M}} F_m = I, \quad (31)$$

such that the probability $P(m)$ is also normalised as

$$\sum_{m \in \mathcal{M}} P(m) = 1. \quad (32)$$

- (ii) In the case of a selective measurement, the sub-ensemble of the systems for which the outcome m has been found to be described by the density matrix

$$\rho'_m = P(m)^{-1}\Phi_m(\rho), \quad (33)$$

where $\Phi_m = \Phi_m(\rho)$ is a positive super-operator, called an operation, and it maps positive operators to positive operators. We also assume that the operation Φ_m obeys the condition

$$\text{tr}\Phi_m(\rho) = \text{tr}\{F_m\rho\}. \quad (34)$$

Equation (34) together with equation (30) yields the normalisation

$$\text{tr}\rho'_m P(m)^{-1} \text{tr}\Phi_m(\rho) = 1. \quad (35)$$

- (iii) The density matrix for the corresponding non-selective measurement is given by

$$\rho' = \sum_{m \in \mathcal{M}} P(m)\rho'_m = \sum_{m \in \mathcal{M}} \Phi_m(\rho), \quad (36)$$

which is normalised according to equations (34) and (31) as

$$\text{tr}\rho' = \sum_{m \in \mathcal{M}} \text{tr}\Phi_m(\rho) = \sum_{m \in \mathcal{M}} \text{tr}\{F_m\rho\} = \text{tr}\rho = 1. \quad (37)$$

An important measurement scheme which can be treated within the framework of the generalised theory of quantum measurements above is the concept of an indirect measurement. Instead of measuring the system we want to measure directly, in an indirect measurement, we perform the measurement on what we will call a quantum probe that has interacted with the system at some point. We may then determine the state of the object we want to measure via correlations with the probe which has resulted from the interaction. An indirect measurement can thus be considered to be consisting of three elements. The three elements are the quantum system to be measured, called the quantum object and has a Hilbert space \mathcal{H}_O , the quantum probe with Hilbert space \mathcal{H}_P and a classical apparatus by which a measurement is performed on the quantum probe following it's interaction with the quantum object. Thus for an ideal measurement, we have three requirements. The first requirement is that prior to the interaction, and at time $t = 0$, the probe is prepared in a well defined quantum state ρ_P while the quantum object is in a state ρ_O . The second requirement is that the the measurement takes place after the interaction is over. So, the interaction between the probe and object may start at time $t = 0$ and end at time $t = \tau > 0$ but the measurement may only take place after the interaction has ended. The third requirement is that the measurement on the probe by the classical apparatus can be described as an ideal measurement by the von Neumann-Lüders projection postulate described above.

At the initial time $t = 0$, the density matrix of the combined system consisting of both probe and object is given by the tensor product $\rho_O \otimes \rho_P$ in the total Hilbert space given by $\mathcal{H} = \mathcal{H}_O \otimes \mathcal{H}_P$. The Hamiltonian of the total system is given by

$$H(t) = H_O + H_P + H_I(t), \quad (38)$$

where H_O and H_P , describe the free evolution of the object and probe respectively. The $H_I(t)$ term describes the evolution due to the interaction between the object and the probe. Outside the interaction time interval, $[0, \tau]$, the term $H_I(t)$ vanishes. The time evolution over this time interval according to the Schrödinger equation can be described by a unitary operator called the time-evolution operator. It is given by

$$U \equiv U(\tau, 0) = T_{\leftarrow} \exp \left[-i \int_0^{\tau} dt H(t) \right], \quad (39)$$

where T_{\leftarrow} is the chronological time-ordering operator, and we have used units such that $\hbar = 1$. If we assume $\rho(0) = \rho_O \otimes \rho_P$ to be the initial density matrix, then in terms of the time-evolution operator, the density matrix at time τ is given by

$$\rho(\tau) = U(\rho_O \otimes \rho_P)U^\dagger. \quad (40)$$

5. The Schrödinger, Heisenberg and Interaction Pictures

The equation of motion for the density matrix is called the Liouville-von Neumann equation and can be derived straightforwardly starting from the Schrödinger equation (1). To see this, consider the initial density matrix

$$\rho(t_0) = \sum_{\alpha} w_{\alpha} |\psi_{\alpha}(t_0)\rangle \langle \psi_{\alpha}(t_0)|. \quad (41)$$

The state of the system at the time t is then given in terms of the time-evolution operator as

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0). \quad (42)$$

We get the Liouville-von Neumann equation by differentiating (42) with respect to time, which is given as

$$\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)]. \quad (43)$$

The Liouville-von Neumann equation (43) can be written more concisely if we introduce the Liouville operator defined as $\mathcal{L}(t)\rho(t) = -i[H(t), \rho(t)]$,

$$\frac{d}{dt}\rho(t) = \mathcal{L}(t)\rho(t). \quad (44)$$

Integrating equation (44), we get the integral form of the Liouville-von Neumann equation, which is given as

$$\rho(t) = T_{\leftarrow} \exp \left[\int_{t_0}^t ds \mathcal{L}(s) \right] \rho(t_0). \quad (45)$$

In the above expressions, the density matrix evolves in time according to the Liouville-von Neumann equation but the operators or observables in the Hilbert space \mathcal{H} is constant. We can obtain an equivalent representation by transferring the time dependence from the density matrix to the operators. The interaction picture density matrix is given by

$$\rho_I(t) \equiv U_I(t, t_0)\rho(t_0)U_I^\dagger(t, t_0). \quad (46)$$

Differentiating this by both sides, the corresponding Liouville-von Neuman equation for the Interaction picture density matrix is given by

$$\frac{d}{dt}\rho_I(t) = -i[H_I(t), \rho_I(t)]. \quad (47)$$

Integrating the differential equation (47), we get the equivalent integral form

$$\rho_I(t) = \rho_I(t_0) - i \int_{t_0}^t ds [H_I(s), \rho_I(s)]. \quad (48)$$

Unlike the for Heisenberg picture operators, the time-evolution of the Interaction picture operators are not generated by the full Hamiltonian H but only by the free part H_0 . If the interaction part is zero, i.e $\hat{H}_I(t) = 0$, then we have $H_0 = H$, such that $U_0(t, t_0) = U(t, t_0)$ and $U_I(t, t_0) = I$ and so the Interaction picture becomes identical to the Heisenberg picture. On the other hand, if $H_0 = 0$, then we have $\hat{H}_I(t) = H(t)$ such that $U_0(t, t_0) = I$ and $U_I(t, t_0) = U(t, t_0)$ so that it becomes identical to the Schrödinger picture.

6. Relativistic Quantum Mechanics

In the relativistic framework, everything is re-formulated in terms of four-vectors with a Minkowski background. Here the four-vector $x^\mu = (x^0, \vec{x})$ represents the space-time coordinates of an event x in Minkowski space-time and we use the symbol $\eta_{\mu\nu}$ to represent the Minkowski metric. The Interaction picture representation of the Schrödinger equation (12) is used with the Interaction Hamiltonian defined as in equation (13), where $\mathcal{H}(t, \vec{x})$ is the Hamiltonian density. In a given fixed coordinate system, the vector $|\Psi(t)\rangle$ gives the state of a quantum mechanical system at each time $x^0 = t$ and so allows the evaluation of expectation values for all observables which are localised on the hypersurface $x^0 = t = \text{constant}$ in Minkowski space-time. In order to get a Lorentz invariant generalisation of this concept, consider a state vector associated with a three-dimensional space like hypersurface σ which is defined as a manifold in Minkowski space that extends to infinity in all directions. Consider further that at each point $x \in \sigma$ on the hypersurface, there exists a unit, timelike normal vector $n^\mu(x)$ satisfying the normalisation

$$n_\mu(x)n^\mu(x) = 1, n^0(x) \geq 1. \quad (49)$$

The state vector then becomes a functional $|\Psi\rangle = |\Psi(\sigma)\rangle$ in the space of all such hypersurfaces. The same is true of the density matrix of the system which is given as the functional $\rho = \rho(\sigma)$. The generalisation of the Schrödinger equation (12) in this relativistic framework is thus a functional differential equation and is given by the Schwinger-Tomonaga equation, which is given by

$$\frac{\delta |\Psi(\sigma)\rangle}{\delta\sigma(x)} = -i\mathcal{H}(x) |\Psi(\sigma)\rangle. \quad (50)$$

A similar generalisation also present the form for the Liouville-von Neumann equation is given by

$$\frac{\delta |\Psi(\sigma)\rangle}{\delta\sigma(x)} = -i [\mathcal{H}(x), \rho(\sigma)]. \quad (51)$$

In direct analogy to partial differential equations, the Schwinger-Tomonaga equation is subject to the integrability condition

$$\frac{\delta^2 \rho(\sigma)}{\delta\sigma(x)\delta\sigma(y)} - \frac{\delta^2 \rho(\sigma)}{\delta\sigma(y)\delta\sigma(x)} = [[\mathcal{H}(x), \mathcal{H}(y)], \rho(\sigma)] = 0, \quad (52)$$

where the points x and y are located on the same hyper surface σ . This integrability condition is a direct consequence of the requirement of the micro causality of the Hamiltonian density which state that $\mathcal{H}(x)$ and $\mathcal{H}(y)$ must commute if x and y are space like separated, i.e $[\mathcal{H}(x), \mathcal{H}(y)] = 0$ for $(x - y)^2 < 0$. This integrability condition insures that the Schwinger-Tomonaga equation has a unique solution $\rho(\sigma)$ once one has chosen an appropriate initial density matrix $\rho(\sigma_0)$ for an initial hypersurface σ_0 . This solution is normally given as

$$\rho(\sigma) = U(\sigma, \sigma_0)\rho(\sigma_0)U^\dagger(\sigma, \sigma_0), \quad (53)$$

where $U(\sigma, \sigma_0)$ is the generalisation of the unitary time-evolution operator given by

$$U(\sigma, \sigma_0) = T_{\leftarrow} \exp \left[-i \int_{\sigma_0}^{\sigma} d^4x \mathcal{H}(x) \right], \quad (54)$$

where T_{\leftarrow} is the chronological time-ordering operator, as usual.

A foliation of Minkowski space is defined as a smooth one-parameter family $\mathcal{F} = \{\sigma(\tau)\}$ of space like hypersurfaces $\sigma(\tau)$ with the property that each space-time point x is located on precisely one hypersurface of the family. A given foliation $\sigma(\tau)$ gives rise to a corresponding family of state vectors $|\Psi(\tau)\rangle = |\Psi(\sigma(\tau))\rangle$. The Schwinger-Tomonaga equation (50) can then be re-formulated as an integral equation

$$|\psi(\tau)\rangle = |\Psi(0)\rangle - i \int_{\sigma_0}^{\sigma(\tau)} d^4x \mathcal{H}(x) |\Psi(\sigma_x)\rangle, \quad (55)$$

where we have denoted $\sigma_x = \sigma(\tau)$ for exactly one parameter value τ .

The hypersurfaces $\sigma(\tau)$ of a foliation can be defined with the help of an implicit equation of the form

$$f(x, \tau) = 0, \quad (56)$$

where $f(x, \tau)$ is a smooth scalar function. With an appropriate normalisation of f , the unit normal vector can be assumed to be given by

$$n_\mu(x) = \frac{\partial f(x, \tau)}{\partial x^\mu}. \quad (57)$$

Consider two infinitesimally separated hypersurfaces corresponding to two parameter values τ and $\tau + d\tau$. Then $d|\Psi(\tau)\rangle = |\Psi(\tau + d\tau)\rangle - |\Psi(\tau)\rangle$, which according to equation (55) is therefore

$$d|\Psi(\tau)\rangle = -i \int_{\sigma(\tau)}^{\sigma(\tau+d\tau)} d^4x \mathcal{H}(x) |\Psi(\tau)\rangle. \quad (58)$$

The four-volume element d^4x can be re-written as $d^4x = d\sigma(x) \left| n_0 \frac{\partial x_0}{\partial \tau} \right| d\tau = d\sigma(x) \left| \frac{\partial f}{\partial \tau} \right| d\tau$. Substituting this into equation (58) and dividing both sides by $d\tau$, we get

$$\frac{d}{d\tau} |\Psi(\tau)\rangle = -i \int_{\sigma(\tau)} d\sigma(x) \left| \frac{\partial f}{\partial \tau} \right| \mathcal{H}(x) |\Psi(\tau)\rangle \equiv -iH(\tau) |\Psi(\tau)\rangle. \quad (59)$$

For a particular example, let's consider an observer O moving along a straight world line $y(\tau) = n\tau$ with constant velocity \vec{v} such that

$$n = \frac{dy}{d\tau} = (\gamma, \gamma\vec{v}), \quad (60)$$

where $\gamma = \frac{1}{\sqrt{1-|\vec{v}|^2}}$. Here we can see that n is the four-velocity of O . Here, the parameter τ is the proper time of the observer O , or the time measured by a clock carried along the world line $y(\tau)$ by O . At each fixed value of τ , the time axis in the rest frame of observer O is in the direction of the unit vector n while the instantaneous three-space at τ is given by the flat, spacelike hypersurface $\sigma(\tau)$ which is orthogonal to n and contains the point $y(\tau)$. So, the function f (56) is then defined as

$$f(x, \tau) \equiv n(x - y(\tau)) \equiv nx - \tau = 0. \quad (61)$$

We see that the hypersurface $\sigma(\tau)$ consists of all the space-time points x with which the observer O assigns the same time coordinate τ . Since $\left| \frac{\partial f}{\partial \tau} \right| = |-1| = 1$, equation (59) then becomes

$$\frac{d}{d\tau} |\Psi(\tau)\rangle = -i \int_{\sigma(\tau)} d\sigma(x) \mathcal{H}(x) |\Psi(\tau)\rangle \equiv -iH(\tau) |\Psi(\tau)\rangle. \quad (62)$$

In the coordinate system where the normal vector n coincides with the time axis, i.e. $n^\mu = (1, 0, 0, 0)$, then equation (62) becomes identical to the Schrödinger equation (12).

On the other hand, the term $\left| \frac{\partial f}{\partial \tau} \right| \neq 1$ in general, equation (62) only applies if the observer O travels along a straight world-line with constant four-velocity n .

Now that the formalism has been established, we proceed to develop it to allow for more general space-times. As discussed in the Abstract, the first step is to extend the formalism based on a straight world line to an accelerated world line. If we allow for acceleration such that $n(\tau)$ varies in time, then we have to include a different value for $\left| \frac{\partial f}{\partial \tau} \right|$ in general. To take another example, let's consider the case of the observer O moving in a uniformly accelerating frame. In other words, the reference frame of the observer O is such that there is a constant non-zero three-acceleration \vec{a} measured from within that frame. If we again assume that the unit normal vector $n(\tau)$ is the four-velocity, then the four-acceleration is given by

$$a^\mu(\tau) = \frac{dn^\mu(\tau)}{d\tau}. \quad (63)$$

Now, since the four-velocity is always normalised as $n^\mu(\tau)n_\mu(\tau) = \eta_{\mu\nu}n^\mu(\tau)n^\nu(\tau) = 1$, we also have $\frac{d}{d\tau}(n_\mu(\tau)n^\mu(\tau)) = 2n_\mu(\tau)a^\mu(\tau) = 0$. Therefore any four-acceleration is orthogonal to the corresponding four-velocity at a given time τ , i.e. $a^\mu(\tau)n_\mu(\tau) = 0$ at all times τ . In the observer frame O , the unit normal vector n coincides with the time axis. Therefore the four-velocity of O is $n^\mu = (1, 0, 0, 0)$ when measured with respect to its own reference frame, while the four-acceleration is given by $a^\mu = (0, \vec{a})$. Noting that the magnitude of the four-acceleration as measured in the observer frame O is $|(0, \vec{a})| = |\vec{a}| = a$, the only way to make sure that the acceleration of O remains uniform is to insure that the magnitude of the four-acceleration remains constant at a . Using the fact that a^μ and n^μ are orthogonal, we obtain for the four-acceleration,

$$a^\mu(\tau) = (a \sinh(a\tau), \frac{a}{|\vec{a}|} \cosh(a\tau)\vec{a}), \quad (64)$$

and for the four-velocity,

$$n^\mu(\tau) = (\cosh(a\tau), |\vec{a}|^{-1} \sinh(a\tau)\vec{a}), \quad (65)$$

such that $a^\mu a_\mu = a^2(\sinh^2(a\tau) - \cosh^2(a\tau)) = -a^2$, $a^\mu n_\mu = 0$ and $n^\mu n_\mu = \cosh^2(a\tau) - \sinh^2(a\tau) = 1$. The observer therefore follows a hyperbolic world-line $y(\tau) = (a^{-1} \sinh(a\tau), a^{-1} |\vec{a}|^{-1} \cosh(a\tau)\vec{a})$ and the function (56) is given by $f(x, \tau) \equiv n(\tau)(x - y(\tau)) = n(\tau)x = 0$. Therefore $\left| \frac{\partial f}{\partial \tau} \right| = |a^\mu(\tau)x_\mu|$ and so the Schwinger-Tomonaga equation (59) for the case of an accelerated world line now becomes

$$\frac{d}{d\tau} |\Psi(\tau)\rangle = -i \int_{\sigma(\tau)} d\sigma(x) |a^\mu(\tau)x_\mu| \mathcal{H}(x) |\Psi(\tau)\rangle \equiv -iH(\tau) |\Psi(\tau)\rangle. \quad (66)$$

7. Conclusion

The main part of this paper provides an overview of a framework for special relativistic quantum mechanics, although it is not full quantum field theory. It describes a statistical interpretation of non-relativistic quantum mechanics for multi-particle systems and then proceeds to use that interpretation to describe a theory of quantum measurement. The framework was then generalised to a framework for special relativistic quantum mechanics. This framework was initially formulated by Breuer and Petruccione [1].

As shown above, we have derived the form of the Schwinger-Tomonaga equation for the case where the observer frame O is moving at constant velocity in special relativity. In our new work, we derived the form in the case of uniform acceleration.

Our next step will be to derive Bell-state measurements within the framework before attempting to introduce more general metrics into the framework other than the Minkowski metric.

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