Quantum measurements along accelerated world-lines

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Introduction

- Working with a framework for relativistic quantum mechanics
- Not full quantum field theory
- Ultimate goal is to extend framework to include curved space times
- First step is to work with accelerated world-lines

Statisitical Interpretation of Quantum Mechanics

- Spectral theorem:
  - For any operator $\hat{R}$, there exists a unique spectral family E_r such that $\hat{R} = \int_{-\infty}^{\infty} r dE_r$, where:
    - $E_{r'} \geqslant E_r$ for $r' > r$.
    - $\lim_{\epsilon \to +0} E_{\epsilon + r} = E_r$.
    - $\lim_{r \to -\infty} E_r = 0$ and $\lim_{r \to +\infty} E_r = I$, where I is the identity.
  - $\{\forall r \mid \left[E_r, \hat{R}\right] = 0\}$ and the domain of $\hat{R}$ is given by $D(\hat{R}) = \{\psi \in \mathcal{H} \mid \int_{-\infty}^{+\infty} d\langle\psi \,| E_r |\, \psi\rangle < \infty\}$, where $\int_{-\infty}^{+\infty} r^2 d\langle\psi \,| E_r |\, \psi\rangle = \int_{-\infty}^{+\infty} r^2 d\|E_r\psi\|^2 = \|\hat{R}\|^2$.
- Consider statistical ensemble $\epsilon = \{S^{(1)}, S^{(2)}, \cdots, S^{(N)}\}$, where $S^{(1)}, S^{(2)}, \cdots S^{(N)}$ are identical quantum systems.
  - Postulate 1: Under certain conditions, a complete characterisation of ensemble is given by state a state vector $|\psi\rangle$ in Hilbert space $\mathcal{H}$.
  - Postulate 2: Measurable quantities in the statistical ensemble are represented by self-adjoint operators in the Hilbert space $\mathcal{H}$.

Statistical mixtures

- Consider M ensembles $\epsilon_1, \epsilon_2, \cdots \epsilon_M$, each described by a normalised state vector $\psi_\alpha, \alpha = 1, 2, \cdots, M$.
  - For total ensemble with weights $w_\alpha \geqslant 0, \sum_{\alpha=1}^{M} w_\alpha = 1$, operator $\hat{R}$ yields cumulative distribution function $F_R(r) = \sum_\alpha w_\alpha \langle \psi_\alpha \left| \hat{R} \right| \psi_\alpha \rangle$ where E_r is the spectral decomposition.
  - For mean value of R, $E(R) = \sum_\alpha w_\alpha \langle \psi_\alpha \left| \hat{R} \right| \psi_\alpha \rangle$
  - Introduce density matrix, $\rho = \sum_\alpha w_\alpha \left| \psi_\alpha \right\rangle \left\langle \psi_\alpha \right|$
- Distribution function can then be written as $F_R = \mathtt{tr}\{E_r \rho\}$

Indirect quantum measurements

An indirect measurement of a quantum system consists of 3 elements:

- The Quantum system to be measured.
- The quantum probe.
- A classical measurement device with which to perform a measurement on the quantum probe after it has interacted with the quantum system.

Schröedinger, Heisenberg and Interaction pictures

- The Schröedinger picture:
  - The state vector $|\psi(t)\rangle$ evolves in time according to the Schröedinger equation $i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$
  - The Schröedinger equation in terms of the time evolution operator $U(t, t_0)$, which evolves the state vector from an initial time to a later time, is given by $i\frac{\partial}{\partial t}U(t, t_0) = H(t)U(t, t_0)$, where the state vector is given by $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$.
  - The time evolution operator is given by $U(t, t_0) = T_{\leftarrow} \exp\left[-i\int_{t_0}^{t} ds H(s)\right]$
  - The density matrix evolves in time according to the equation $\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] = \mathcal{L}(t)\rho(t)$
- The Heisenberg picture is obtained by transferring the time dependence from the density matrix to the operators in the Hilbert space.
  - The Heisenburg picture is related to the Schröedinger picture by $A_H(t) = U^{\dagger}(t, t_0)A(t)U(t, t_0)$, where $A(t)$ is an arbitrary observable.
- The Heisenberg and Schröedinger pictures are limiting cases of the Interaction picture
  - The Hamiltonian can be written as $H(t) = H_0 + \hat{H}_I(t)$

Measurements in Relativistic Quantum Mechanics

- Consider $x^\mu = (x^0, \vec{x})$
  - The state vector is then associated with a 3-dimensional spacelike hypersurface σ as a manifold on Minkowski space-time where at each point $x \in \sigma$ there's a normal unit vector satisfying $n_\mu(x) n^\mu(x) = 1, n^0 \geqslant 1$
  - The state vector is then given by the functional $|\Psi\rangle = |\Psi(\sigma)\rangle$
  - Similarly, the density matrix is given by the functional $\rho = \rho(\sigma)$
- The Schwinger-Tomonaga equation:
  - The Schwinger-Tomonaga equation is given by $\frac{\delta|\Psi(\sigma)\rangle}{\delta\sigma(x)} = -i\mathcal{H}(x)\,|\Psi(x)\rangle$
  - Or for the density matrix $\frac{\delta\rho(\sigma)}{\delta\sigma(x)} = [\mathcal{H}(x), \rho(\sigma)]$
  - If the foliation $\sigma(\tau)$ gives rise to a corresponding family of state vectors $|\Psi(\tau)\rangle = |\Psi(\sigma(\tau))\rangle$ then the Schwinger-Tomonaga equation can be re-written as $|\Psi(\tau)\rangle = |\Psi(0)\rangle - i\int_{\sigma_0}^{\sigma(\tau)} d^4x\mathcal{H}(x)\,|\Psi(\sigma_x)\rangle$

Foliations on Minkowski Space-Time

- The hyper surfaces $\sigma(\tau)$ of a foliation can be defined with the help of a function $f(x,\tau) = 0$
  - With an appropriate normalisation of f, the normal vector $n_\mu(x)$ at the point $x \in \sigma(\tau)$ is given by $n_\mu(x) = \frac{\partial f(x,\tau)}{\partial x^\mu}$.
  - The Schwinger-Tomonaga equation is then $\frac{d}{d\tau} |\Psi(\tau)\rangle = -i \int_{\sigma(\tau)} d\sigma \left| \frac{\partial f}{\partial \tau} \right| \mathcal{H}(x) |\Psi(\tau)\rangle \equiv -iH(\tau) |\Psi(\tau)\rangle$

Measurements along straight world-lines

- If a measurement is along a world-line $y(\tau) = n\tau$ with constant speed v, then $f(x,\tau) \equiv nx - \tau = 0$ and the 4-velocity of observable O is $n = \frac{dy}{d\tau} = (\gamma, \gamma\vec{v})$ with $\gamma = \frac{1}{\sqrt{1-|\vec{v}|}}$.
- The instantaneous 3-space at fixed τ is given by the hypersurface $\sigma(\tau)$, orthogonal to n and contains the point $y(\tau)$, which is define by $n(x - y(\tau)) \equiv nx - \tau = 0$.
- For constant velocity of observer O, $\left|\frac{\partial f}{\partial \tau}\right| = 1$, therefore the Schwinger-Tomonaga equation becomes $\frac{d}{d\tau}\left|\Psi(\tau)\right\rangle = -i\int_{\sigma(\tau)} d\sigma(x)\mathcal{H}(x)\left|\Psi(\tau)\right\rangle \equiv -iH(\tau)\left|\Psi(\tau)\right\rangle$.

Measurements along accelerated world-lines

- In general, for an accelerated world line $y(\tau)$, $f(x,\tau) \equiv n(\tau)(x - y(\tau)) = 0$
  - The instantaneous 3-space at fixed τ is again given by the hypersurface $\sigma(\tau)$, orthogonal to n.
  - The term $\left|\frac{\partial f}{\partial \tau}\right| \neq 1$ in general, so it must be taken into account.
- Consider the case of uniform acceleration a with the 4-acceleration given as $a(\tau) = \frac{dn}{d\tau}$:
  - $\frac{d}{d\tau}\left[n(\tau)\right]^2 = 2n_\mu(\tau)a^\mu = 0$ since $n_\mu n^\mu = 1$, so $n_\mu(\tau)a^\mu(\tau) = 0$ for all n and a.
  - Therefore, world line follows a hyperbolic path with $y(\tau) = a^{-1}\sinh(a\tau)\hat{t} + a^{-1}\cosh(a\tau)\hat{x}$ and $n(\tau) = \cosh(a\tau)\hat{t} + \sinh(a\tau)\hat{x}$, so $n(\tau)y(\tau) = 0$ and $\left|\frac{\partial f}{\partial \tau}\right| = |a(\tau)x|$.
  - The Schwinger-Tomonaga equation is then $\frac{d}{d\tau}\left|\Psi(\tau)\right\rangle = -i\int_{\sigma(\tau)} d\sigma(x)\,|a(\tau)x|\,\mathcal{H}(x)\left|\Psi(\tau)\right\rangle \equiv -iH(\tau)\left|\Psi(\tau)\right\rangle$

Conclusion and future plans

- Able to do measurements by observers moving along accelerated world-lines.
- Will try to calculate measurements of Bell states with current formalism.
- Will try to work with metrics other than the Minkowski metric to extend formalism to curved space-time.