Multi-fluid Perturbations of Cosmology in $f(T)$-gravity Model

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Talk's Plan

Why gravity?

"It's the gravity that shapes the large scale structure of the universe, even though it is the weakest of four categories of forces" Stephen Hawking.

Introduction of the theory of gravity

The physics of gravity is completely changed after the development of general theory of relativity

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{C^4}T_{\mu\nu} \quad (1)$$

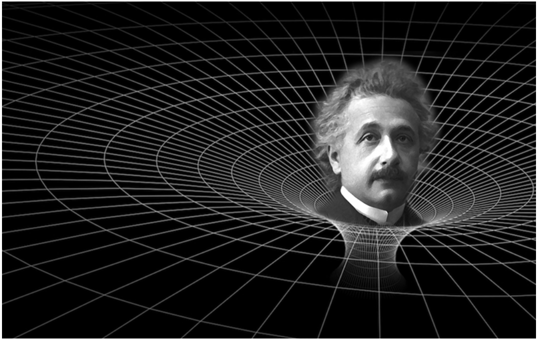

Figure 1: Space-time curvature

Introduction Cosmology

- It deals about origin of the universe, the formation large-scale structures and dynamics, and the ultimate fate of the universe.

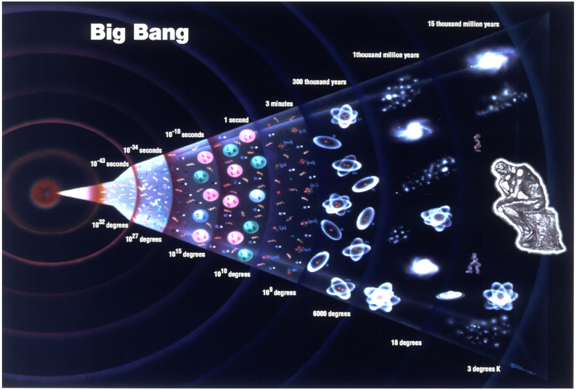

Figure 2: A history of the Universe. Credit: grandunificationtheory.com

- Standard model of cosmology
  1. Isotropic
  2. Homogeneous
- **Cosmological probs**: Dark matter, , Inflationary Universe ....
- **Why the modified theory of gravity is needed?**

Modified theory of gravity

- The standard model of cosmology so-called ΛCDM model is going to be a big challenge due to the presence of the most extremely dramatic behavior of the universe.
- For instance, the discovery of dark energy and dark matter still becomes a puzzle in physics.
- Dark energy responsible for current accelerating expansion universe.
- To solve such kind of probes in cosmology, the modification of the general theory of gravity count as an alternative approach to explain the acceleration expansion universe without dark energy scenarios.
- There are many modified theories of gravity, such as:
  - Gauss Bonnet $f(G)$, G-being the universal gravitational constant,
  - $f(R)$, R-being scalar curvature ,
  - $f(T)$, T-being torsion scalar and $f(T,B)$, B-being the boundary term .

$f(T)$-gravity

- In this talk, the second-order gravity so called $f(T)$-gravity only considered.
- In GR, torsion T is assumed to vanish and in **teleparallel gravity theory**, curvature R is assumed to vanish.
- Fortunately, the two basic theories of gravity describe the gravitational interaction equivalently TEGR.
- So, torsion is an alternative direction of describing the gravitational field interaction the energy-momentum tensor is the source in both theories the source of the curvature in GR and the source of torsion in teleparallel gravity theory
- One of the basic differences from the usual internal gauge models in many ways, the most significant being the presence of a tetrad field.

- In teleparallel gravity theory, we use the new connection which is Weitzenböck connection instead of the usual connection affine or Levi-Civita connection, and we use the torsion scalar T to describe the Lagrangian density rather than curvature scalar R.
- Ricci scalar yields as

$$R = T + 2\nabla^a T_{ab}{}^{b} = T + 2\nabla^a T_a \; , \tag{2}$$

$$= T + B \; , \tag{3}$$

where $T_a = T_{ab}{}^{b}$ and $B = 2\nabla^a T_a$ is the boundary term.

- Then Eq. (3) gives us a clear hint on how we can explore the relationship between R and T.

Universe Compositions

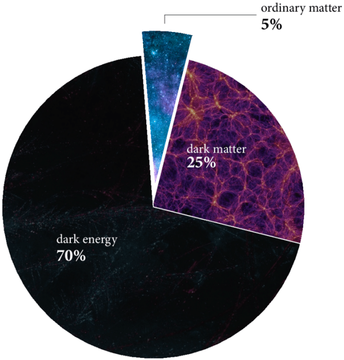

Figure 3: Energy content of the universe according to the standard model of particle physics and general relativity. Sources: Max-Planck-Institute for Astrophysics Garching and Pixabay

- Our universe is composed of many fluids namely, radiation, baryonic matter, dark matter, etc... and in most cases they interact with each other.
- In this work, we consider the growth of matter density with cosmic time if the universe has multi-fluid systems.
- In this assumptions the Universe has filled by i^{th}-component non-interacting fluids and the total energy-momentum tensor of the fluid becomes the sum of the individual spices,

$$\Theta_{ab}^{total} \equiv \sum_i \Theta_{ab}^i = \rho_i u_a^i u_b^i + p_i h_{ab}^i + q_a^i u_b^i + q_b^i u_a^i + \pi_{ab}^i \quad (4)$$

where ρ_i is the energy density, p_i is the pressure, q_i is the heat flux and π_i is the anisotropic stress tensor of the i^{th} fluids.

- So far, the four-velocity vector is either energy frame $u^i_a = u^i_{aE}$ or particle frame $J^i_a = J^i_{aN} = 0$ for component fluids [1].
- Then, the inhomogeneity variables the spatial gradient of gauge invariant quantities in i^{th}-component fluids express as follows:

$$D^i_m = \frac{a}{\rho^i_m}\tilde{\nabla}_a\rho^i_m\ , \qquad Z_a = a\tilde{\nabla}_a\theta\ , \tag{5}$$

$$Y^i_m = \tilde{\nabla}_a p^i_m\ , \qquad \epsilon^i_a = \frac{a}{p^i_m}\frac{\partial p^i_m}{\partial\rho^i_m}\tilde{\nabla}_a s^i\ . \tag{6}$$

The physical parameter ϵ_a, is the dimensionless quantities that represent the entropy perturbations for i^{th}-component fluids. The expression of this dimensionless parameter with pressure can be expressed as [1]

$$p\epsilon_a = \sum_i p_i \epsilon_a^i + \frac{1}{2}\sum_{i,j}\frac{h_i h_j}{h}(c_{si}^2 - c_{sj}^2)S_a^{ij} \,, \tag{7}$$

where $w_i = \frac{p_i}{\rho_i}$ is equation of state parameter and $c_{is}^2 = \frac{\partial p_i}{\partial \rho_i}$ is sound speed for the i^{th}-component fluids. The conservation equation for the i^{th} components of non-interacting fluids as [1, 2]

$$\dot{\rho}_i + h_i\theta + h_i\tilde{\nabla}^a\Psi_a^i = 0 \,, \qquad \text{since} \qquad h_i = \rho_i + p_i \,, \tag{8}$$

and

$$h_i\dot{u}_a + \tilde{\nabla}_a p_i + h_i\dot{\Psi}_a^i - (3c_{si}^2 - 1)\frac{\theta}{3}\Psi_a^i + \Pi_a^i = 0 \,, \tag{9}$$

where

$$\Psi^i_a = \frac{q^i_a}{h_i} + V^i_a, \qquad \Pi^i_a = \tilde{\nabla}^b \pi^i_{ab} \,, \tag{10}$$

the term V^i_a is the relative velocity of the i^{th} component fluids which is relative to the fundamental observer.
For the case of perfect fluids, the heat flux and anisotropic stress are going to zero and $\Psi^i_a = V^i_a$. The relative velocity of the i^{th} component fluid to the fundamental observer O_u can be defined as

$$V^i_a \equiv u^i_a - u_a \,. \tag{11}$$

The detailed analysis of cosmological perturbations in multi-component fluids is done for GR approach [3, 1] and $f(R)$ approach [2].

- In the present work we extend this investigation in $f(T)$ gravitational theory and we discuss the cosmological perturbations for multi-fluid systems and their cosmological implications.
- Then, the general form of the linear evolution equations for total fluids in $f(T)$ gravity theory by $1+3$ covariant formalism is presented in Eqs. (5)- (9) as:

$$\dot{D}^m_a = -(1+w)Z_a + \theta w D^m_a + w\theta\varepsilon_a \,, \tag{12}$$

$$\dot{Z}_a = \Bigg[-\left(\frac{3\rho_m}{2f'} + \frac{3w}{2f'}\rho_m\right) + \Big[\frac{\theta^2}{3} + \frac{1}{2f'}(1+3w)\rho_m$$
$$+\frac{f''}{f'}\dot{T}\theta - \tilde{\nabla}^2\Big]\frac{c_s^2}{1+w}\Bigg] D^m_a$$
$$-\Big[\frac{\rho_m f''}{2f'^2} + \frac{3wf''}{2f'^2}\rho_m + \frac{f'''}{f'}\dot{T}\theta + \frac{f''^2}{f'^2}\dot{T}\theta\Big]\mathcal{F}_a - \frac{f''}{f'}\theta\mathcal{B}_a$$
$$-\left(\frac{f''\dot{T}}{f'} + \frac{2\theta}{3}\right)Z_a$$
$$+\Big[\frac{\theta^2}{3} + \frac{1}{2f'}(1+3w)\rho_m + \frac{f''}{f'}\dot{T}\theta - \tilde{\nabla}^2\Big]\frac{w\varepsilon_a}{1+w} \,, \tag{13}$$

$$\dot{\mathcal{F}}_a = \mathcal{B} - \frac{\dot{T}}{1+w}(c_s^2 D^m_a + w\varepsilon_a) \,, \tag{14}$$

$$\tag{15}$$

Evolution equations for component fluid

- Another key points in a multi-fluid system are the matter component and velocity fluctuations.
- These equations are useful to study the growth of energy density fluctuations for i^{th}-component fluids. Then the first-order spatial gradient equation for matter component fluid from Eq. (5) and velocity fluctuations from Eq. (9) respectively

$$\dot{D}^i_a = -(1+w_i)Z_a + \frac{1+w_i}{1+w}\theta\left(c_s^2 D^m_a + w\varepsilon\right) - \theta w_i D^i_a - \theta c^2_{si} D^i_a$$
$$-\theta w\varepsilon - (1+w_i)a\tilde{\nabla}_a\tilde{\nabla}_b V^b_i \ , \qquad (16)$$

$$\dot{V}^i_a - (3c^2_{si} - 1)\frac{\theta}{3}V^i_a = -\frac{1}{ahh_i}\left(\left(hc^2_{si}D^i_a\rho_i + hp_i\varepsilon_a\right) - h_i c^2_s D_a\rho - h_i p\varepsilon_a\right) \qquad (17)$$

Applied some techniques

Scalar Decomposition techniques

$$\Delta_m = a\tilde{\nabla}^a D^m_a\ , \qquad Z = a\tilde{\nabla}^a Z_a\ , \qquad \mathcal{F} = a\tilde{\nabla}^a \mathcal{F}_a\ , \qquad \mathcal{B} = a\tilde{\nabla}^a \mathcal{B}$$

$$S_{ij} = a\tilde{\nabla}^a S^{ij}_a \qquad V_{ij} = a\tilde{\nabla}^a V^{ij}_a \qquad \Delta^i_m = a\tilde{\nabla}^a D^i_a \qquad V_i = a\tilde{\nabla}^a V^i_a$$

Harmonic Decomposition Technique

$$X = \sum_k X^k(t)Q^k(\vec{x})\ , \ \text{ and } \ Y = \sum_k Y^k(t)Q^k(\vec{x})\ ,$$

where k is the wave-number and $Q^k(x)$ is the eigenfunctions of the covariant derivative. Wave-number k represent the order of harmonic oscillator and relate with the scale factor as $k = \frac{2\pi a}{\lambda}$, where λ is the wavelength of the perturbations.

2^{nd} order evolution equation

$$\ddot{\Delta}^k_m = -\left(\frac{f''\dot{T}}{f'} + (\frac{2}{3} - c_s^2)\theta\right)\dot{\Delta}^k_m + \Bigg[\left(\frac{3\rho_m}{2f'} + \frac{3w\rho_m}{f'}\right)(1+w)$$
$$-\frac{2\theta^2}{3}(c_s^2 - w) - \frac{c_s^2}{f'}(1+3w)\rho_m + \frac{f''}{2f'}\dot{T}\theta(c_s^2 + 2w) - \frac{k^2}{a^2}c_s^2\Bigg]\Delta^k_m$$
$$+\left[\frac{\rho_m f''}{2f'^2} + \frac{3wf''}{2f'^2}\rho_m + \frac{f'''}{f'}\dot{T}\theta + \frac{f''^2}{f'^2}\dot{T}\theta\right](1+w)\mathcal{F}^k + \frac{f''\theta}{f'}(1+w)$$
$$+\left[\frac{\theta^2}{3} - \frac{1}{2f'}(1+3w)\rho_m + \frac{2f''}{f'}\dot{T}\theta - \frac{k^2}{a^2}\right]w\epsilon^k + \frac{f''\theta}{f'}\dot{T}c_s^2\epsilon^k \quad (1$$

$$\ddot{\mathcal{F}}^k = \frac{\dddot{T}}{\dot{T}}\mathcal{F}^k - \frac{2\ddot{T}}{1+w}c_s^2\Delta^k_m - \frac{\dot{T}}{1+w}c_s^2\dot{\Delta}^k_m - \frac{2\ddot{T}}{1+w}w\epsilon^k - \frac{\dot{T}}{1+w}w\dot{\epsilon}^k\,, \quad (1$$

2^{nd} order evolution equation

$$\ddot{\Delta}^k_i = \Big[\Big(\frac{3\rho_m}{2f'} + \frac{3w}{2f'}\rho_m\Big)(1+w_i) + \Big[\frac{2\theta^2}{3} + \frac{1}{2f'}(1+3w)\rho_m + \frac{2f''}{f'}\dot{T}\theta -$$

$$\frac{k^2}{a}\Big]\frac{c_s^2(1+w_i)}{1+w}\Big]\Delta^k_m - \Big[\Big(\frac{f''\theta\dot{T}}{f'} + \frac{2\theta^2}{3}\Big)w_i - (w_i - c_{si}^2)\Big(\frac{\theta^2}{3} + (1+3w)\frac{\rho_{m}}{2f'}$$

$$+ \Big(\frac{f''\theta\dot{T}}{f'} + \frac{2\theta^2}{3}\Big)\frac{c_s^2}{1+w_i}\Big]\Delta^k_i - \Big[\Big(\frac{f''\dot{T}}{f'} + \frac{2\theta}{3}\Big) - \frac{1+w_i}{1+w}\theta c_s^2\Big]\dot{\Delta}^k_m$$

$$+ \Big[\frac{\rho_m f''}{2f'^2} + \frac{3wf''}{2f'^2}\rho_m + \frac{f'''}{f'}\dot{T}\theta + \frac{f''^2}{f'^2}\dot{T}\theta\Big](1+w_i)\mathcal{F}^k + \frac{f''}{f'}\theta(1+w_i)\dot{\mathcal{F}}^k$$

$$+ \Big[\frac{f''\dot{T}\theta}{f'}\frac{(1+w_i)c_s^2}{1+w} - \Big(\frac{f''\theta\dot{T}}{f'} + \frac{2\theta^2}{3}\Big)w + \Big(\frac{f''\theta\dot{T}}{f'} + \frac{2\theta^2}{3}\Big)\frac{1+w_i}{1+w}w\Big]\epsilon^k$$

$$+ \Big(\frac{f''\dot{T}}{f'} + \frac{2\theta}{3}\Big)(1+w_i)\frac{k^2}{a}V^i - \frac{k^2}{a^2}\frac{w(1+w_i)\epsilon}{1+w},$$

$$\ddot{V}^k_{ij} - (3c^2_{si} - 1)\frac{\dot{\theta}}{3}V^k_{ij} - (3c^2_{si} - 1)\frac{\theta}{3}\dot{V}^k_{ij} - (c^2_{si} - c^2_{sj})\dot{\theta}V^k_j - (c^2_{si} - c^2_{sj})\theta\dot{V}^k_j =$$
$$\frac{\theta}{3a}\left[c^2_{si}S^k_{ij} + (c^2_{si} - c^2_{sj})\frac{\Delta^k_j\rho_j}{h_j} + \frac{1}{h_i}p_i\epsilon^k - \frac{1}{h_j}p_j\epsilon^k\right] - \frac{1}{a}\Big[c^2_{si}\dot{S}^k_{ij} +$$
$$(c^2_{si} - c^2_{sj})\frac{\Delta^k_j\dot{\rho}_j}{h_j} + (c^2_{si} - c^2_{sj})\frac{\dot{\Delta}^k_j\rho_j}{h_j} - (c^2_{si} - c^2_{sj})\frac{\Delta^k_j\rho_j\dot{h}_j}{h^2_j} + \frac{1}{h_i}\dot{p}_i\epsilon^k$$
$$+\frac{1}{h_i}p_i\dot{\epsilon}^k - \frac{\dot{h}_i}{h^2_i}p_i\epsilon^k - \frac{1}{h_j}\dot{p}_j\epsilon^k - \frac{1}{h_j}p_j\dot{\epsilon}^k + \frac{\dot{h}_j}{h^2_j}p_i\epsilon^k\Big],$$
$$\ddot{S}^k_{ij} = -\frac{\theta k^2}{3a^3}V^k_{ij} + \frac{k^2}{a^3}\dot{V}^k_{ij}\,.$$

Growth of matter in Radiation-Dominated Universe

- In this regard, we found the solution of the second-order component equations (19) and (20) to analysis the growth of energy density in i^{th} component fluids.
- Indeed, they are many component fluids in the entire Universe and in this work, we also consider Universe has radiation and dust with torsion fluid are the major components of non-interact fluids.

Radiation Dominated Universe

The solution for radiation dominated universe in short wavelength limit

$$\Delta_d^k(z) = C_1\left(1+z\right)^{\frac{-296\,n^2+433\,n-72}{288\,n^2-360\,n}}\text{BesselJ}\left(\frac{\iota}{576\,n^2-720\,n}, \frac{2}{3}\,\frac{\sqrt{3}\pi}{\lambda\,\left(1+z\right)^2}\right)$$

$$C_2\left(1+z\right)^{\frac{-296\,n^2+433\,n-72}{288\,n^2-360\,n}}\text{BesselY}\left(\frac{\iota}{576\,n^2-720\,n}, \frac{2}{3}\,\frac{\sqrt{3}\pi}{\lambda\,\left(1+z\right)^2}\right).$$

where $\iota^2 = 39616\,n^5 + 165376\,n^4 - 2282560\,n^3 + 3048913\,n^2$ $-1156752\,n + 5184$

Short-Wavelength modes

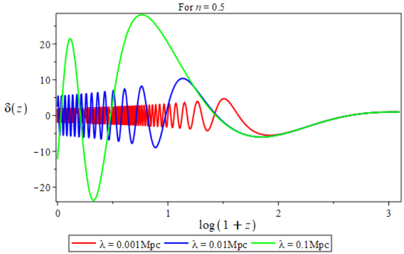

Figure 4: The growth of dust fluctuations $\delta(z)$ versus z for Eq. (20) in the radiation dominated Universe, for short-wavelength mode at $n = 0.5$ and for different wavelength λ

Short-Wavelength modes

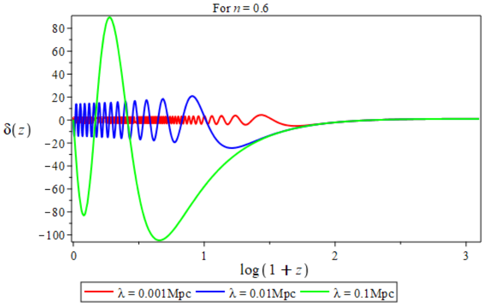

Figure 5: The growth of dust fluctuations $\delta(z)$ versus z for Eq. (20) in the radiation dominated Universe, for short-wavelength mode at $n = 0.6$ and for different wavelength λ

Long-Wavelength limit

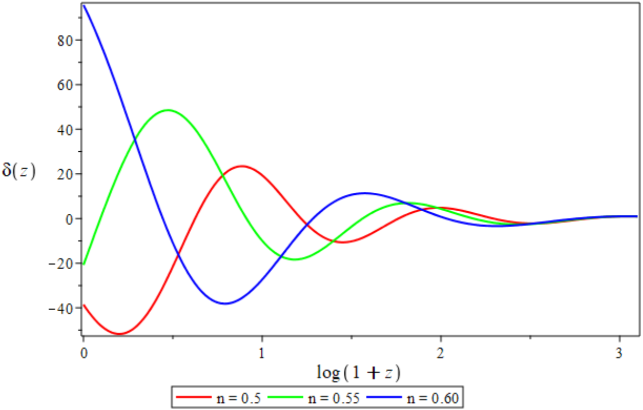

Figure 6: The growth of dust fluctuations $\delta(z)$ versus z for Eq. (**??**) in the radiation dominated Universe for n is closer to 0.5

Long-wavelength limit

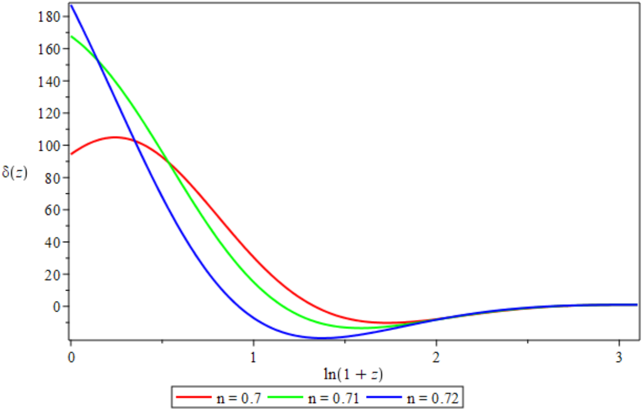

Figure 7: The growth of dust fluctuations $\delta(z)$ versus z for Eq. (**??**) in the radiation dominated Universe for long-wavelength mode for different n values.

Conclusions

- The multi-fluid system, the perturbations evolution equations are examined,
- The scalar and harmonic decomposition techniques are applied.
- We apply the quasi-static approximation is implemented to examine the growth of the density contrast with cosmic-time.
- We further investigated the growth dust density contrasts in the radiation epoch.
- From the results, we conclude that the contributions of torsion is large enough for the formation of large-scale structure.


Thank you

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