Black holes and nilmanifolds: Quasinormal modes as the fingerprints of extra dimensions?

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Einstein-Hilbert action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(R - 2\Lambda + \mathcal{L}_m \right)$$

Einstein's grav. constant:	$\kappa = 8\pi G c^{-4}$
determinant:	$g = \det(g_{\mu\nu})$
Ricci scalar:	$R = g^{\mu\nu} R_{\mu\nu}$
cosmological constant:	$\Lambda = 3/L^2$
Lagrangian for the matter fields:	$\mathcal{L}_m$

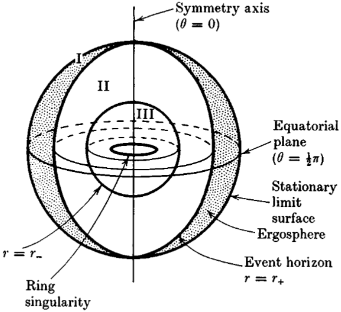
Einstein-Hilbert action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\,(R - 2\Lambda + \mathcal{L}_m)$$

$$\Big\Downarrow \; \delta S = 0$$

Einstein field equations for *flat space*, *in vacuum*:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \cancel{\Lambda g_{\mu\nu}} = \cancel{\kappa T_{\mu\nu}}$$



Hawking & Ells, *The Large Scale Structure of Space-time*

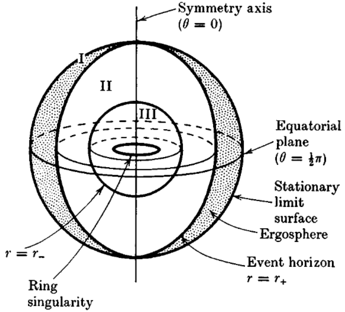
The "no-hair" conjecture

All stationary black hole solutions in GR can be completely characterised by three independent, externally observable, and classical parameters:

- *mass M,*
- *electric charge Q,*
- *angular momentum a.*

Stationary, neutral, spherically-symmetric black hole:

$$g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



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The "no-hair" conjecture

$$f(r) = 1 - \frac{2M}{r}$$

event horizon:	$r_H = 2M$
photon orbit:	$r_c = 3M$
length scale:	$M = Gm^{BH}c^{-2} = 1$
$(G = c = 1)$	*Bekenstein*

1957: Regge and Wheeler's grav. perturbations

The birth of black hole perturbation theory:

L REVIEW VOLUME 108, NUMBER 4 NOVEMBER

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

JOHN A. WHEELER, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\
g_{\mu\nu} \to g'_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \qquad (g_{\mu\nu} \gg h_{\mu\nu})
\end{aligned}
$$

Quasinormal mode: "ringdown"

Quasinormal mode and frequency

$$\Psi(x^\mu) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\theta, \phi) \;, \qquad \omega_{sn\ell} = \omega_R - i\omega_I$$

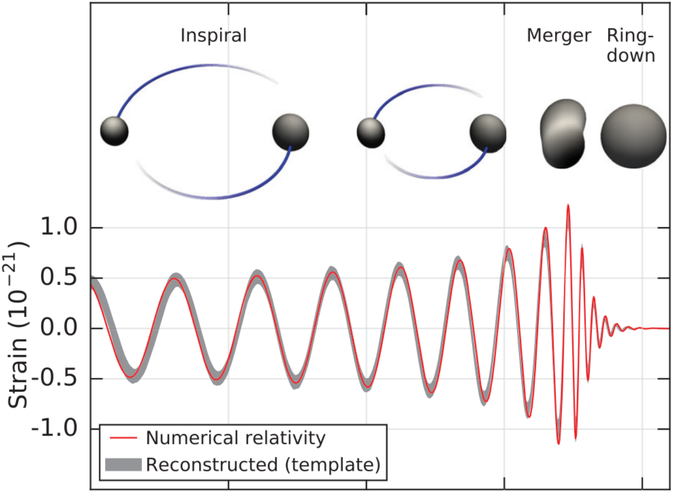
- ⋆ $\mathbb{R}e\{\omega\}$ = physical oscillation frequency
- ⋆ $\mathbb{I}m\{\omega\}$ = damping → dissipative, "quasi"

Quasinormal mode and frequency

$$\Psi(x^\mu) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\theta, \phi) \; , \qquad \omega_{sn\ell} = \omega_R - i\omega_I$$

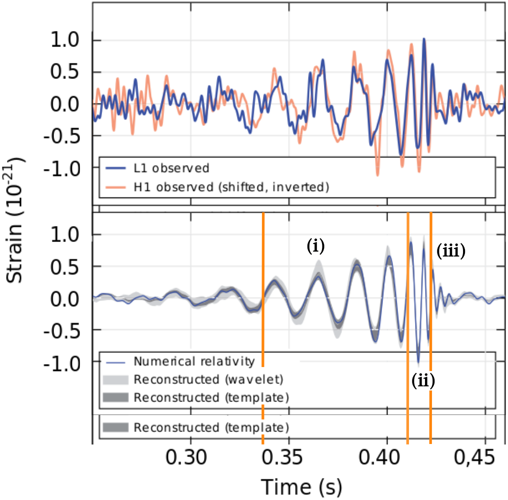
- ⋆ s: spin of perturbing field
- ⋆ m: azimuthal number for spherical harmonic decomposition in θ_i
- ⋆ ℓ: angular/multipolar number for spherical harmonic decomposition in θ, ϕ
- ⋆ n: overtone number labels QNMs by a monotonically increasing $|\mathbb{I}m\{\omega\}|$

Quasinormal mode: "ringdown"



B. P. Abbott *et al.*, PRL **116**, 061102 (2016)

Quasinormal mode: "ringdown"



(i) inspiral
(ii) merger
(iii) ringdown

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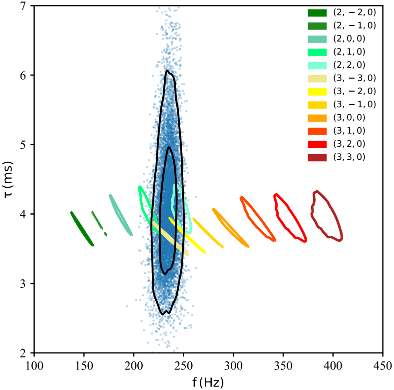
the *fundamental* $(\ell, m, n) = (2, 2, 0)$ *mode* dominates ringdown



a multimodal analysis of the GW150914 data using PyRing, see Carullo *et al.*

Quasinormal mode: "ringdown"

Stationary, neutral, spherically-symmetric black hole:

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Due to symmetries, only 2 ODEs needed:

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Due to symmetries, only 2 ODEs needed:

- ⋆ *Angular behaviour encapsulated by spheroidal harmonics:*

$$\nabla^2 Y^s_{m\ell}(\theta, \phi) = -\frac{\ell(\ell+1)}{r^2} Y^s_{m\ell}(\theta, \phi)$$

- ⋆ *s.t. QNM computations depend on radial behaviour*

Black hole wave equation:

$$\frac{d^2}{dr_*^2}\varphi(r_*) + \left[\omega^2 - V(r)\right]\varphi(r_*) = 0 \, , \qquad \frac{dr}{dr_*} = f(r)$$

$\rightarrow$ reduces to a second-order ODE in r

Black hole wave equation:

$$\frac{d^2}{dr_*^2}\varphi(r_*) + \left[\omega^2 - V(r)\right]\varphi(r_*) = 0\,, \qquad \frac{dr}{dr_*} = f(r)$$

$\rightarrow$ subjected to **QNM boundary conditions**

purely ingoing: $\varphi(r_*) \sim e^{-i\omega(t+r_*)}$ $\quad r_* \rightarrow -\infty\ (r \rightarrow r_H)$

purely outgoing: $\varphi(r_*) \sim e^{-i\omega(t-r_*)}$ $\quad r_* \rightarrow +\infty\ (r \rightarrow +\infty)$

Waves escape domain of study at the boundaries $\Rightarrow$ dissipative

What is the significance of these extra dimensions?

Today: *A path to physics BSM?*

- 1980s: “KK renaissance”, 1984 “superstring revolution”
- 1998: Arkani-Hamed, Dimopoulos, Dvali: large EDs
- 1999: Randall and Sundrum: warped EDs

...

Why the negativity?

Negatively-curved EDs: a BSM landscape of untapped potential?

Phenomenological implications:

- natural resolution to the hierarchy problem

  $\rightarrow$ volume grows exponentially with ℓ_G/ℓ_c

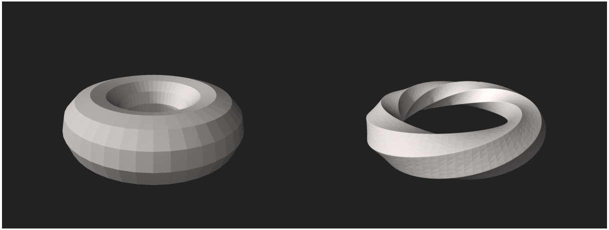
  $\rightarrow$ RSI-like KK mass spectrum w/o light KK modes

- zero modes of Dirac operator emerges w/o gauge breaking
- enables homogeneity and flatness of observed universe

Any Lie group G of dimension d can be understood as a d-dimensional differentiable manifold. To compactify solvable G, we quotient by the lattice Γ. For nilpotent groups, the resultant twisted torus is a nilmanifold.

$$[Z_b, Z_c] = f^a{}_{bc}\, Z_a \;, \quad f^a{}_{bc} \; = -f^a{}_{bc}$$

$$\mathcal{R} = -\frac{1}{4}\delta_{ad}\,\delta^{be}\,\delta^{cg} f^a{}_{bc} f^d{}_{eg}$$



Wikimedia Commons, Torus & Twisted torus

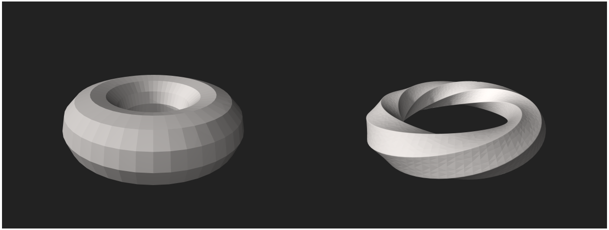
Consider the Heisenberg algebra...

$$[Z_1, Z_2] = -\mathtt{f} Z_3 \ , \quad [Z_1, Z_3] = [Z_2, Z_3] = 0$$

$$\mathrm{d}e^3 = \mathtt{f} e^1 \wedge e^2 \ , \quad \mathrm{d}e^1 = 0 \ , \quad \mathrm{d}e^2 = 0$$

$$e^1 = r^1 dy^1 \ , \quad e^2 = r^2 dy^2 \ , \quad e^3 = r^3 (dy^3 + N r^1 dy^2) \ , \quad N = \frac{r^1 r^2}{r^3} \mathtt{f}$$

...gives us the metric $ds^2_{\mathcal{N}} = g^{\mathcal{N}}_{ij} dx^i dx^j$



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An unconventional QNM application

How can we exploit available GW searches to investigate extra-dimensional scenarios?

The BH-nilmanifold metric

$$ds^2_{7D} = g^{BH}_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu + g^{\mathcal{N}_3}_{ij}(\mathbf{y})dy^i dy^j$$

$$\Psi^s_{n\ell m}(t,r,\theta,\phi,y_1,y_2,y_3) = \sum_{n=0}^{\infty}\sum_{\ell,m}\frac{\psi_{sn\ell}(r)}{e^{i\omega t}r}Y^s_{m\ell}(\theta,\phi)Z(y_1,y_2,y_3)$$

$$ds^2_{BH} = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(\sin^2 d\theta^2 + d\phi^2)$$

$$f(r) = 1 - 2M/r$$

$$ds^2_{\mathcal{N}_3} = r_1^2 dy_1^2 + r_2^2 dy_2^2 + r_3^2(dy_3 + Nr_1 dy_2)^2$$

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Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{\mathcal{N}_3}\right) \sum \Phi(\mathbf{x}) Z_k(\mathbf{y}) = 0 \ ,$$

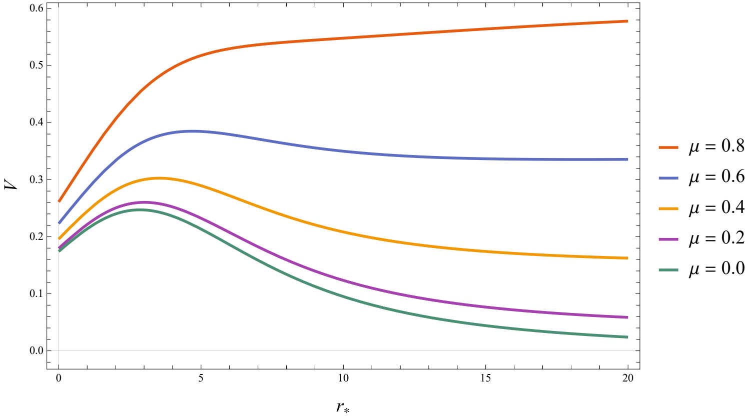
$$\nabla^2_{\mathcal{N}_3} Z_k(\mathbf{y}) = -\mu^2_k Z_k(\mathbf{y})$$

The wavelike equation

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\psi = 0$$

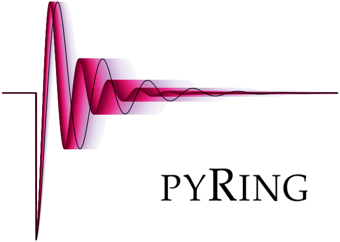
$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

The fundamental mode: $n = 0, \ell = 2$



Agnostic bounds from ringdown GWs

LVK's *Tests of GR with binary black holes from* GWTC-1, GWTC-2, GWTC-3



a python package for ringdown analysis

$$\delta\omega = \omega^{GR}(1 + \delta\omega)$$

$$\delta\tau = \tau^{GR}(1 + \delta\tau)$$

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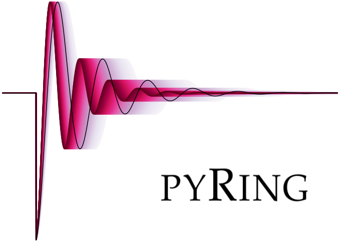

a python package for ringdown analysis

$$\delta\omega_{O3} = 0.02^{+0.07}_{-0.07}$$

$$\delta\tau_{O3} = 0.13^{+0.21}_{-0.22}$$

Conclusions

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- ⋆ Detecting modifications to GR is considered to be beyond the sensitivity of modern detectors (still model-agnostic, first-order, etc.)

- ⋆ Here, we have determined a QNM detectability bound on extra dimensions

  ↪ roadmap for new model-agnostic extra dimensions search

Thank you

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