Analytic Continuation of the Generalized Epstein zeta function for calculating finite system corrections in ϕ^4 theory

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Abstract. Previously an equation of state for the relativistic hydrodynamics encountered in heavy-ion collisions at the LHC and RHIC has been calculated using the QCD trace anomaly obtained with lattice methods. This lattice calculation extrapolates the trace anomaly to an infinite system, which neglects the possible finite system size corrections that should be present in heavy ion collisions. In order to check whether the corrections are indeed negligible at the scales encountered at these experiments, it is necessary to understand how these corrections arise in quantum field theories. We construct a massive ϕ^4 theory with periodic boundary conditions on n of the 3 spatial dimensions. $2 \to 2$ NLO scattering is then computed, while analytically verifying unitarity remains intact. Due to needing to employ a novel regularization method dubbed denominator regularization, it was crucial to derive an analytic continuation to the Generalized Epstein zeta function, which is expected to have applications in further studying finite size corrections to field theories using denominator regularization.

1. Introduction

In Heavy Ion Colliders such as at the LHC and RHIC, heavy-ions (such as Pb nuclei) are collided at nearly the speed of light ($\gamma \gg 10$ Lorentz factor). In these heavy ion collisions [1, 2] there is an apparent formation of Quark Gluon Plasma (QGP) [3], where the correlations between the outgoing low-momentum particles appear to be well described by nearly inviscid relativistic hydrodynamics. This calculation uses an Equation of State (EoS) provided by a lattice QCD calculation that is extrapolated to infinite system size [4].

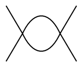
It is currently unclear what happens in QCD just above the transition temperature $T = 180$ MeV. There is strong evidence of a second order phase transition, but the nature of the new phase is unknown. While for large temperatures asymptotic freedom requires that the phase is essentially an ideal gas, the more complicated behaviour discussed above is found at the relatively small (and experimentally accessible) temperatures of $T \sim 350$MeV. It is therefore necessary to understand how reliably the behaviour found in the finite systems (such as heavy ion or parton collisions) can be extrapolated to effectively infinite systems, such as the QGP found in the ~ 0.000001 seconds after Big Bang. Understanding and modeling this phase is therefore not only of interest in high energy physics but also in cosmology, in particular in studies of the early universe.

The dependence of the low viscosity on the lattice QCD calculation of the EoS brings the underlying assumptions of the calculation under scrutiny. A possible erroneous assumption to be investigated is that heavy ion collisions can be well approximated as infinite sized systems [5]. Indeed quenched lattice QCD calculations have shown significant possible corrections dependent on the size of the system [6]. An analytic derivation of the finite size effects on the equation of state (or equivalently the trace anomaly) is therefore sought. This work is a step in that direction, with the intention to develop and understand the mathematical techniques necessary for a full treatment necessary for finite temperature finite sized QCD.

2. Finite Sized ϕ^4 Theory

Let us consider the ϕ^4 Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

in a system with periodic boundary conditions. If we consider n compact spatial (and 1 infinite temporal) dimensions, with the ith dimension being parameterized by $[-\pi L_i, \pi L_i]$ with periodic boundary conditions (or equivalently identifying the endpoints $\pm\pi L_i$). This discretizes the possible spatial momenta to $\vec{p} = (\frac{k_1}{L_1}, \frac{k_2}{L_2}, \ldots, \frac{k_n}{L_n},)$ where $\vec{k} \in \mathbb{Z}^n$. In analogy with [7] we can define $(-i\lambda)^2 iV(p^2) \equiv$  with p being the total incoming momentum. One then finds [8] in $n = 3$ spatial dimensions that up to NLO

$$V(p^2, \{L_i\}) = -\frac{1}{2}\int_0^1 dx \int \frac{d\ell_E^0}{2\pi} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{(2\pi)^3 L_1 L_2 L_3} \frac{1}{[\ell_E^2 + \Delta^2]^2} \tag{2}$$

where $\Delta^2 \equiv -x(1-x)p^2 + m^2 - i\varepsilon$ and $\ell_E^\mu = (\ell_E^0, \frac{k^i}{L_i} + x\, p^i)^\mu$. Using denominator regularization as introduced in [8] we can regularize 2 using some $\epsilon > 0$ as follows:

$$V(p^2, \{L_i\}) = -\frac{1}{2}\int_0^1 dx \int \frac{d\ell_E^0}{2\pi} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{(2\pi)^3 L_1 L_2 L_3} \frac{\mu^\epsilon}{[\ell_E^2 + \Delta^2]^{2+\frac{\epsilon}{2}}}. \tag{3}$$

We now safely perform the ℓ_E^0 integral to get

$$V(p^2, \{L_i\}) = -\frac{1}{2}\int_0^1 dx \frac{1}{(2\pi)^4 L_1 L_2 L_3} \frac{\sqrt{\pi}\,\Gamma\left(\frac{3+\epsilon}{2}\right)}{2\Gamma\left(\frac{4+\epsilon}{2}\right)} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{\mu^\epsilon}{[\sum_{i=1}^3(\frac{k_i}{L_i} + x\, p^i)^2 + \Delta^2]^{\frac{3+\epsilon}{2}}}. \tag{4}$$

The sum at the end of Equation 4 can be related back to the generalized Epstein zeta function, so we will need to find some useful analytically continued form of it in order to regularize Equation 4 properly. It is noted that the use of denominator regularization and the generalized Epstein zeta function should not be confuse with zeta function regularization, which is a distinct regularization procedure.

3. Generalized Epstein zeta Function

We have seen above that we require an analytic continuation of the generalized Epstein zeta function [9], which would allow us to isolate the divergence in Equation 4. Let us then explicitly define

$$\zeta(\{a_i\}, \{b_i\}, c; s) \equiv \sum_{\vec{n}\in\mathbb{Z}^p} \left[a_i^2 n_i^2 + b_i n_i + c\right]^{-s}, \tag{5}$$

where repeated indices are implicitly summed over. We can now employ the Poisson Summation formula over an n-dimensional lattice Λ^n with lattice dual Λ^{*n}, which is given by

$$\sum_{\vec{m}\in\Lambda^n} f(\vec{m}) = \frac{1}{\sqrt{\det(\Lambda^n)}} \sum_{\vec{k}\in\Lambda^{*n}} F(\vec{k}), \tag{6}$$

where F is the Fourier transform given by $F(\vec{k}) \equiv \int d^n m \, \exp(2\pi i\vec{k}\cdot\vec{m}) f(\vec{m})$. Here we are interested in

$$f(\vec{m}) = \left[a_i^2 m_i^2 + b_i m_i + c - i\varepsilon\right]^{-s}, \tag{7}$$

with some $\varepsilon > 0$ is introduced to ensure no poles are integrated over. We can then calculate (where all suppressed indices of sums and products runs from $i = 1$ to $i = n$):

$$F(\vec{k}) = \int d^n m \, \exp(2\pi i\vec{k}\cdot\vec{m}) \left[a_i^2 m_i^2 + b_i m_i + c - i\varepsilon\right]^{-s} \tag{8}$$

Now by shifting $x_i = m_i + \frac{b_i}{2a_i^2}$ and $c' = c - \sum_{i=1}^{p} \frac{b_i^2}{4a_i^2}$ we get

$$F(\vec{k}) = \exp\left(-2\pi i \sum \frac{k_i b_i}{2a_i^2}\right) \int d^n x \, \exp(2\pi i\vec{k}\cdot\vec{x}) \left[a_i^2 x_i^2 + c' - i\varepsilon\right]^{-s} \tag{9}$$

We can now see that that integral almost looks like the Fourier transform of some radial function over a lattice with lattice spacings a_i, so we can actually use the formula for a Fourier transform of a radial function[10] to get

$$F(\vec{k}) = \frac{\exp(-2\pi i \sum \frac{k_i b_i}{2a_i^2})}{\prod a_i} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty dr \, r^{n-1} [r^2 + c' - i\varepsilon]^{-s} \, {}_0F_1\left(;\frac{n}{2}; -\pi^2 r^2 \sum \frac{k_i^2}{a_i^2}\right). \tag{10}$$

While this might look intimidating, this integral (with ${}_0F_1$ being the generalized hypergeometric function) can be evaluated using software such as Mathematica to get (valid for $\mathrm{Re}(s) > \frac{n}{2}, n > 1, c' \in \mathbb{R}, \varepsilon > 0, \|\vec{k}\| \neq 0$)

$$F(\vec{k}) = \frac{\exp(-2\pi i \sum \frac{k_i b_i}{2a_i^2})}{\prod a_i} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\pi \sum \frac{k_i^2}{a_i^2}}{c' - i\varepsilon}\right)^{\frac{s-2n}{4}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(s)} \mathrm{K}_{\frac{n-2s}{2}}\left(2\pi\sqrt{\sum \frac{k_i^2}{a_i^2}}\sqrt{c' - i\varepsilon}\right) \tag{11}$$

$$F(\vec{k}) = \frac{2\pi^s}{\prod a_i \Gamma(s)} \exp\left(-2\pi i \sum \frac{k_i b_i}{2a_i^2}\right) \left(\frac{\sum \frac{k_i^2}{a_i^2}}{c' - i\varepsilon}\right)^{\frac{s-2n}{4}} \mathrm{K}_{s-\frac{n}{2}}\left(2\pi\sqrt{(c' - i\varepsilon)\sum \frac{k_i^2}{a_i^2}}\right), \tag{12}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind. For the $\vec{k} = \vec{0}$ case we can either very carefully take the limit, or redo the above calculation with $\vec{k}$ identically $\vec{0}$. Both yield

$$F(\vec{0}) = \frac{\pi^{\frac{n}{2}}}{\prod a_i} \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)} (c' - i\varepsilon)^{\frac{n-2s}{2}} \tag{13}$$

Since we were careful with keeping track of the lattice spacings of the dual lattice, we can now plug our results straight into the Poisson summation formula to get (after expanding $c' = c - \sum \frac{b_i^2}{4a_i^2}$)

$$\sum_{\vec{m}\in\mathbb{Z}^n} \left[a_i^2 m_i^2 + b_i m_i + c - i\varepsilon\right]^{-s} = \frac{\pi^{\frac{n}{2}}}{\prod a_i} \frac{\Gamma\left(s-\frac{n}{2}\right)}{\Gamma(s)} (c - \sum \frac{b_i^2}{4a_i^2} - i\varepsilon)^{\frac{n-2s}{2}} + \frac{2\pi^s}{\prod a_i \Gamma(s)} \times \tag{14}$$

$$\sum_{\vec{k}\in\mathbb{Z}^n}{}' \exp\left(-2\pi i \sum \frac{k_i b_i}{2a_i^2}\right) \left(\frac{\sum \frac{k_i^2}{a_i^2}}{c - \sum \frac{b_i^2}{4a_i^2} - i\varepsilon}\right)^{\frac{s-2n}{4}} \mathrm{K}_{s-\frac{n}{2}}\left(2\pi \sqrt{(c - \sum \frac{b_i^2}{4a_i^2} - i\varepsilon) \sum \frac{k_i^2}{a_i^2}}\right),$$

which agrees with previous results [8]. Note that $\sum'$ above denotes that it does no sum over $\vec{0}$.

4. Amplitude and Unitarity

We can now substitute Equation 14 into Equation 4 which gives in $n = 3$

$$V(p^2, \{L_i\}) = -\frac{1}{32\pi^2} \int dx \Big[\frac{2}{\epsilon} - 1 + \ln\left(\frac{\mu^2}{\Delta^2}\right) \tag{15}$$
$$+ 2\sum_{\vec{k}\in\mathbb{Z}^3}{}' \cos\left(2\pi x \sum k_i p^i L_i\right) K_0\left(2\pi\sqrt{\Delta^2 \sum (k_i L_i)^2}\right)\Big].$$

Then using a modified $\overline{\text{MS}}$ subtraction scheme, we can get the renormalized

$$\overline{V}(p^2, \{L_i\}) = -\frac{1}{32\pi^2} \int dx \Big[\ln\left(\frac{\mu^2}{\Delta^2}\right) \tag{16}$$
$$+ 2\sum_{\vec{k}\in\mathbb{Z}^3}{}' \cos\left(2\pi x \sum k_i p^i L_i\right) K_0\left(2\pi\sqrt{\Delta^2 \sum (k_i L_i)^2}\right)\Big].$$

Here we can recognize the ln term as corresponding to the standard result in infinite ϕ^4 systems [7]. As one would then expect the second term in the integral vanishes in the limit as all $L_i \to \infty$ since $\lim_{x\to\infty} K_0(x) = 0$. We can also see that we can reduce the effective number of finite dimensions by quite simply taking the corresponding $L_i \to \infty$, since only terms in the sum with the corresponding $k_i = 0$ will survive the limit. We can then find that, as in the infinite system case,

$$\mathcal{M} = \lambda\left[1 + \lambda\left(\overline{V}(s) + \overline{V}(t) + \overline{V}(u)\right)\right] \tag{17}$$

up to NLO, with s, t and u being the Mandelstam variables. The optical theorem then requires that the total cross-section

$$\sigma_{\text{tot}} = 2\text{Im}(\mathcal{M}). \tag{18}$$

By then setting $L_i = L$ and introducing $R = L\sqrt{\frac{s}{4} - m^2}$, one can then find that

$$\sigma_{tot} = \frac{\lambda^2}{16\pi} \frac{\pi^{\frac{1-n}{2}}}{\Gamma\left(\frac{3-n}{2}\right)} \frac{1}{L\sqrt{s}} \sum_{0\leq l<R^2}^{*} \frac{r_n(l)}{\sqrt{R^2 - l}^{n-1}} \tag{19}$$

and

$$2\text{Im}(\mathcal{M}) = \frac{\lambda^2}{16\pi} \frac{1}{L\sqrt{s}} \frac{1}{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3-n}{2}\right)} \sum_{0\leq l<R^2}^{*} \frac{r_n(l)}{\sqrt{R^2 - l}^{n-1}}, \tag{20}$$

showing that the amplitude respects the optical theorem, and equivalently our S-matrix is therefor unitary.

5. Conclusion

The analytic continuation of the generalized Epstein zeta function presented here was shown to give a self-consistent theory up to NLO in finite sized ϕ^4 theory. It is hugely valuable for regularizing and renormalizing in this finite system, and one can expect that it will be in other finite sized field theories as well. We have therefore shown a derivation of the formula, shown it leads to self-consistent results (and limits to the standard infinite system result) and shown how it is necessary in the above calculation.

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