Compact stars: Numerical solutions to the structure equations using Python

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Abstract

The study of compact stars is a topic very valuable for the testing of modern physics in order to better understand the behaviour of cold dense nuclear matter. Compact stars have no fusion processes occurring within them. The only way these stars are then capable of supporting themselves is through the degeneracy pressure of the fermions that constitute these objects. These stars can then be modelled as a degenerate Fermi gas of either electrons or neutrons. This study aimed to solve for the Newtonian and Tolman-Oppenheimer-Volkoff (TOV) structure equations through a numerical approach using Python in order to model the behaviour of these stars. White dwarfs were modelled as a Fermi gas of electrons while the neutron star was modelled first as a pure neutron gas and then as a mix of neutrons,protons and electrons. It was found that within certain limits, the results obtained particularly for the neutron stars, compared well to expected values for the mass of these objects in literature.

1 Introduction

Through fusion processes, stars combine hydrogen and helium to form heavier elements. Eventually all hydrogen and helium in the core is burnt up and the core consists largely of Carbon or Oxygen. As a result, nuclear fusion processes stop and the star's temperature decreases. The star begins to shrink in size as the core pressure increases. If the star's mass is sufficiently large enough, the star will be able to reinitialise fusion processes to form heavier elements. The smaller the mass of the star, the more the core of the star has to be contracted for the required heat to start fusion processes. If the star's original mass is below 8 solar masses then the gravitational pressure is too weak to reach the required density and temperature to begin Carbon fusion. The outer regions of the star expand due to the high core temperature and are blown away by stellar winds. What remains is a core composed mainly of Carbon and Oxygen. This resulting object is a white dwarf with no fusion processes taking place. Through the degeneracy pressure of the electrons, the star is able to maintain hydro-static equilibrium [1].

Similarly, neutron stars are stabilised by the degeneracy pressure of neutrons. If the initial mass of the star is greater than 8 solar masses, carbon and oxygen fusion is now possible.

Around this core of heavier elements, are layers of hydrogen and helium that also undergo fusion. As long as these fusion processes produce the required heat for fusion, fusion to heavier elements will continue in the core along with the fusion of lighter elements in the outer shell. However, once Iron is produced in the core, no more fusion processes accompanied with the release of energy will occur in the core. This iron core then increases in mass while its radius shrinks as there is insufficient thermal energy that is produced to support its structure. At some critical mass, the iron core collapses. The density increases so much that electron capture occurs to form neutrons out of protons and neutrons. The atomic nuclei become more and more neutron rich with increasing density. Eventually, at some critical density, the nuclei are unable to bind neutrons and the neutrons drop out of the nuclei to form a neutron liquid that surrounds the nuclei. If the density is even greater, there will just be a dense in-compressible core of neutrons with a small number of protons and electrons present. The collapse is a rapid and the outer layers fall and bounce off the in-compressible, core leaving behind the remnant neutron star.The study of these stars is very valuable as they provide a means of studying the behaviour of dense, cold nuclear matter in conditions that are close to the density of or even exceeding the density of atomic nuclei [3]. Currently, the EoS of neutron stars is not known. The only way we can then study them is through comparision of generated models and observed data[2]

1.1 Structure equation for a polytrope

The structure of a star is maintained through a balancing act between the outward pressure and the inward gravitational pull. The structure equations describing this balancing act are given by $\frac{dm}{dr} = \rho(r)4\pi r^2$ and $\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$

Rewriting these equations in terms of energy density we find

$$\frac{dm}{dr} = \frac{\epsilon(r)4\pi r^2}{c^2} \tag{1}$$

$$\frac{dp}{dr} = -\frac{G\epsilon(r)m(r)}{c^2 r^2} \tag{2}$$

These coupled equations need to be solved to determine the relationship between mass, pressure and radius. Because these equations are coupled, by making use of an equation of state, an equation describing the relationship between pressure and energy density, it becomes possible to numerically solve these equations. The distribution of an ideal gas in equilibrium is given by the Fermi-Dirac statistic[5]

$$f(E) = \frac{1}{e^{(E-\mu)/k_B T} + 1} \tag{3}$$

Where E is Energy, μ is the chemical potential, k_B and T the temperature. Let $n_i = \frac{dN}{dV}$ be the number density of particles. For a degenerate ideal Fermi gas (meaning $T \to 0$), the Fermi-Dirac statistic can only either be 1 or 0. By ignoring electrostatic interactions, the number density for a degenerate Fermi gas is given by $n_e = \frac{k_F^3}{3\pi^2\hbar^3}$. Given that the mass density of the star in terms of nucleon mass is given by $\rho = n_e m_n \frac{A}{Z}$ with A and Z being the atomic mass and atomic number, we find that the Fermi momenta of the constituent particles is given by $k_F = \hbar \left(\frac{3\pi^2 \rho Z}{m_n A}\right)^{1/3}$.

The total energy density of the star is given by $\epsilon = n m_N \frac{A}{Z} c^2 + \epsilon_i(k_F)$ with i being the particle species. The energy of the particle species is given by $E(k) = \sqrt{k^2c^2 + m_e^2 c^4}$. The energy density contributions due to the Fermi momenta and the pressure of a system with an isobaric distribution of pressure is given by[1]

$$\epsilon_i = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} \sqrt{k^2c^2 + m_i^2 c^4} k^2 dk \tag{4}$$

$$p = \frac{1}{3} \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} kvk^2 dk \tag{5}$$

Where $v = \frac{kc^2}{E} = \frac{kc^2}{\sqrt{k^2c^2 + m_e^2 c^4}}$. Simplifying these integrals based on whether the fermions are moving relativistically or not, we find a simple relation between energy density and pressure. For both cases we find that $p \approx K_{non-rel} \epsilon^{5/3}$ and $p \approx K_{rel} \epsilon^{4/3}$.

Where $K_{non_rel} = \frac{\hbar^2}{15\pi^2 m_e} \left(\frac{3\pi^2 Z}{m_N c^2 A}\right)^{5/3}$ and $K_{rel} = \frac{\hbar c}{12\pi^2} \left(\frac{3\pi^2 Z}{m_N c^2 A}\right)^{4/3}$ which correspond to the non-relativistic and relativistic cases respectively. The above equations are called polytropes which refers to an equation of state that can be expressed as $p = K\epsilon^\gamma$. Due to the formulation of energy density and pressure we can dimensionless quantities $\bar{\epsilon}$ and $\bar{p}$ such that $p = \epsilon_0 \bar{p}$ and $\epsilon = \epsilon_0 \bar{\epsilon}$ where ϵ_0 has the dimensions of energy density. We can then rewrite our polytrope in dimensionless form as $\bar{p} = \bar{K}\bar{\epsilon}^\gamma$. This allows us to rewrite the structure equations in dimensionless form as

$$\frac{dp}{dr} = -\frac{GM_\odot}{c^2} \frac{\bar{p}^{1/\gamma} \bar{M}(r)}{r^2 \bar{K}^{1/\gamma}} = -\frac{\alpha \bar{p}^{1/\gamma} \bar{M}(r)}{r^2} \tag{6}$$

$$\frac{\bar{M}}{dr} = \frac{4^2 \bar{p}^{1/\gamma} \epsilon_0}{M_\odot c^2 \bar{K}^{1/\gamma}} = \beta r^2 \bar{p}^{1/\gamma} \tag{7}$$

The new defined constants α and β have dimensions km^- and km^{-3} respectively. Both differential equations carry dimension km^-. This means these equations need to be integrated with respect to r. α is given by $\epsilon_0 = \left[\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^\gamma\right]^{1-\gamma}$ where $R_0 = \frac{GM_\odot}{c^2}$.
For very compact stars (like neutron stars), one has to take into account the effect of space-time curvature[4]. These effects become more pronounced as the quantity $2GM/Rc^2$ approaches unity. This star then needs to be described using Einstein's equation $G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$. For an isobaric, static, general relativistic ideal fluid sphere (a star is essentially a fluid sphere) at hydro-static equilibrium, we make use of the Tolman-Oppenheimer-Volkoff (TOV) equation [1]

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \left[1 + \frac{p(r)}{\epsilon(r)}\right] \left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right] \left[1 - \frac{2GM(r)}{c^2 r}\right]^{-1} \tag{8}$$

This equation is already in dimensionless form and strengthens gravitational effects.

1.2 Results

In order to solve the dimensionless structure equations, we make use of Ordinary Differential Equation (ODEs) Solvers using Python. This finds a numerical solution to these equations. For this project the **odeint** function from the **scipy.integrate** library was used to solve the polytropic structure equations. Through some elementary coding it is then possible to create plots using the output from the **odeint** function. These plots give an idea of how the solution behaves and are useful for the physical interpretation of the solution. It is also very easy to extract the pressure, mass and radius of the stars that are solutions to these structure equations. They are simply the maximum values in the arrays for pressure, mass and radius.

In order to use the ODE solvers in Python, we need to choose an initial central pressure that makes physical sense for the situation. Secondly, in the dimensionless structure equations, constants α and β are present.We also have the liberty of choosing the values of these constants. Note that the choice of α affects the value of ϵ_0 which in turn determines the value of β.

1.2.1 White Dwarfs non-relativistic case

In this case, the constituent electrons aren't moving relativistically. This means the polytropic index is given by $\gamma = 5/3$. For this case we expect lower pressure values. As a result we expect the star to support less mass than the relativistic case. We choose that $\alpha = 0.05km$, implying $\epsilon_0 = 2.488 \times 10^{37} erg.cm^{-3}$ and $\beta = 0.005924km^{-3}$. The results for a range of central pressures are as follows

$p_0 = 10^{-15}$	radius = $10500km$	M = $0.39422M_\odot$
$p_0 = 10^{-16}$	radius = $13500km$	M = $0.197582M_\odot$
$p_0 = 10^{-17}$	radius = $16500km$	M = $0.09900736M_\odot$

Table 1: mass and radius for range of non-relativistic central pressure values

1.2.2 White Dwarfs relativistic case

In this case the electrons are moving relativistically. In such a case, the star can support a larger mass but requires a greater central pressure. This large central pressure "squeezes" the electrons into relativistic speeds. In this case the polytropic index is given by $\gamma = 4/3$. Here we chose that $\alpha = 1.473km$, making $\epsilon_0 = 7.463 \times 10^{39} ergs/cm^3 = 4.17M_\odot c^2/km^3$, and $\beta = 52.46km$. The results from the ODE solver are

1.2.3 Neutron star: Full relativistic case

To set up an equation of state that works for a range of Fermi momenta values, we can write the energy density of the star as a function of pressure. This is given by

$p_0 = 10^{-15}$	radius $= 8500km$	mass $= 1.2469M_\odot$
$p_0 = 10^{-16}$	radius $= 13000km$	mass $= 1.2456M_\odot$
$p_0 = 10^{-17}$	radius $= 19900km$	mass $= 1.2357M_\odot$

Table 2: mass and radius for range of relativistic central pressure values

$$\bar{\epsilon}(p) = A_{non-rel}\bar{p}^{3/5} + A_{rel}\bar{p} \tag{9}$$

Where $A_{non-rel}$ and A_{rel} are constants. The first term dominates in low pressures while the second will dominate in high pressures. We redefine and fix ϵ_0 as $\epsilon_0 = \frac{m_n^4 c^5}{(3\pi^2\hbar)^3} = 5.346 \times 10^{36} ergs.cm^{-3} = 0.003006M_\odot c^2.km^{-3}$. Using the equations derived earlier for pressure and energy density, Mathematica can be used to fit the constants $A_{non-rel}$ and A_{rel} over a range of Fermi momentum values. These constants were then found to be $A_{non-rel} = 2.4216$ and $A_{rel} = 2.8663$. With the new ϵ_0, we chose that $\alpha = R_0 = 1.476km$ we obtain that $\beta = 0.03778$. Now because neutron stars are sufficiently compact and massive enough, one needs to consider relativistic effects. We then need to solve for the TOV equation. For comparision we solved both the Newtonian formulism and the TOV equations. The results for a starting central pressure of 0.01 (which is clearly relativistic) is

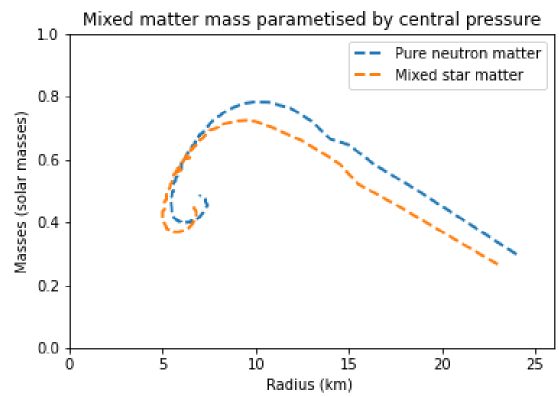
Newtonian	radius $= 11km$	mass $= 1.5312M_\odot$
TOV	radius $= 9km$	mass $= 0.77M_\odot$

Table 3: mass and radius for range of relativistic central pressure values

At this point, it becomes interesting to perform a long string of calculations to create a plot of mass vs radius parameterized by central pressure. By appending each solution for each central pressure value to this array, one can the simply instruct Python to plot the given array.

1.2.4 Neutron stars with protons and electrons

Neutron stars realistically don't only consist of neutrons. This is because free neutrons are unstable and undergoes weak decay. A free neutron has a half-life of about 15 minutes before decaying into a proton, electron and electron anti-neutrino.Note that all the decay products (with the exception of the neutrino) are all Fermions. This means that at some point, all the lower energy levels of the protons and electrons of the system become filled up. Due to Pauli's exclusion principle, no more protons and electrons can then be added to the system. Once in this state, equilibrium between the rates of electron capture and neutron decay is reached stabilising the number of protons and electrons present in the star.We can perform a very similar analysis as done previously. The total energy density and pressure is simply the sum of the energy and pressure contributions from each particle species that is present. We then can write $\epsilon_{tot} = A_{non-re}\bar{p}_{tot}^{3/5} + A_{rel}\bar{p}_{tot}$. Performing the same fitting procedure on Mathematica,we find $A_{non-rel} = 2.572$ and $A_{rel} = 2.891$. Performing a long string of calculations by varying central pressures, we can create a plot demonstrating how the solutions behave. The plot shown includes a plot generated via the same analysis for the pure neutron star for comparision purposes.



2 Discussion and conclusion

In literature, the mass range for neutron stars ranges from about 1 solar mass up until 3[2]. In this study, the masses for the neutron stars lie within this mass range. Secondly, we find interestingly enough comparable results between the pure neutron star and a neutron star with protons and neutrons. The behaviour of the solutions to the equations for a range of central pressures is the same. Note that in this study, no inter-particle interactions were considered. As a future work, these interactions will be included in the model. Secondly, with recent developments in the field of Gravitational waves, another means of probing the equation of state is provided. By studying the wave forms produced by merging neutron stars it becomes possible to determine the properties of the neutron star through studying the effect of tidal forces on the generated wave forms.

References

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