Cavity QED based Open Quantum Walks

Ayanda Zungu[1], **Ilya Sinayskiy**[2,3], **Francesco Petruccione**[2,3,4]

[1]Centre for Space Research, North-West University, Mahikeng, 2745, South Africa
[2]Quantum Research Group, School of Chemistry and Physics, University of KwaZulu-Natal Durban, 4001, South Africa
[3]National Institute for Theoretical and Computational Sciences (NITheCS), South Africa
[4]School of Data Science and Computational Thinking, Stellenbosch University, Stellenbosch 7600, South Africa

E-mail: [1]`arzngu@gmail.com`

Abstract. In this work, we propose a possible experimental scheme for the implementation of open quantum walks (OQWs). The scheme is based on a model consisting of a weakly coupled atom-field system in the dispersive regime inside a high-Q resonator ($Q \sim 10^{12}$). This setup implements an OQW on the line with a two-level atom (driven by a laser) playing the role of the "walker" and the Fock states of the cavity mode as lattice sites of the OQW. The master equation for this system is solved analytically using generating functions for the zero temperature case. Furthermore, the dynamics of the observables are presented for various parameters and their behavior corresponds to the usual physical dynamics, which display collapses and revivals in the evolution of the system observables.

1. Introduction

Realistic quantum systems are open, and therefore any physical system describing their behavior must always include the unavoidable interaction with the environment [1]. The study of open quantum system (OQS) dynamics has recently become crucial, as quantum technologies are approaching regimes where decoherence and dissipation play an important role. For instance, physical systems may experience phase loss and energy exchange with the surroundings, because of this, the quantum effects are suppressed and the system evolves in a non-unitary fashion [1,2].

To address dissipation and decoherence in unitary quantum walks (UQWs) [3,4], which have been used as a basic tool for quantum algorithms, a new type of non-unitary QWs called open QWs was introduced with the aim of incorporating OQS [5]. OQWs are formulated as quantum Markov chain on lattices or graphs. Mathematically, they are described by completely positive trace-preserving (CPTP) maps [1,6] on graphs. The CPTP maps correspond to some dissipative processes which are driving the transition between the nodes of a graph. Unlike UQWs which uses quantum interference effects [3, 4, 7], in OQWs, the transitions between the nodes are strictly driven by the interaction with the environment. Thus, the effects of the environment play a crucial role in the time evolution of OQWs.

Furthermore, it has been suggested that OQWs are capable of performing dissipative quantum computation and to create complex quantum states [5, 8]. The complete description of the framework of OQWs can be found in [5] and a recent article [9] reviews progress on this topic. More importantly, [10] suggested a quantum optics implementation of OQWs, and then derived an OQWs based on the microscopic system-environment model [11].

In this paper, we propose a cavity quantum electrodynamics (QED) based implementation of OQWs. The proposed scheme consist of a single two-level atom interacting with a single cavity mode in a non-resonant fashion driven by the external field. A similar model for the microscopic maser was suggested in [12]. In the proposed cavity QED scheme, a two-level atom plays the role of the "walker" and the Fock states of the cavity mode correspond to the lattice sites of the OQW. We derive the master equation for this system and construct the analytic expressions for the populations using the generating functions for the zero temperature case.

This paper is organized as follows: In Sec. 2 we introduce the model and derive the effective master equation. Then, we apply a specific unitary operator to the effective master equation to include the classical laser driving. Sec. 3 contains the analytical solution for the generalized master equation and discussions; and in Sec. 4 we summarize the results of this paper.

2. Model

We consider the interaction between a two-level atom (qubit) and a quantized single mode field. The interaction Hamiltonian within the rotating wave approximation (RWA) of this system is given by ($\hbar = 1$) [13], $\hat{H}_{\text{int}} = \Delta \hat{a}^\dagger \hat{a} + g\big(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-\big)$, and the dynamics for the system under Born-Markov approximation are given by the Lindblad master equation (ME) [14],

$$\frac{d}{dt}\hat{\rho}(t) = -\imath[\hat{H}_{\text{int}}, \hat{\rho}] + \gamma\left(\mathrm{n_{th}}+1\right)\mathcal{L}\left[\hat{\sigma}_-, \hat{\sigma}_+\right]\hat{\rho} + \gamma \mathrm{n_{th}}\mathcal{L}\left[\hat{\sigma}_+, \hat{\sigma}_-\right]\hat{\rho}, \tag{1}$$

where $\Delta = \omega_f - \omega_a$ stands for the frequency detuning between the field and the qubit, g is the dipole interaction strength, $\hat{\sigma}_\pm$ are the Pauli raising and the lowering operators for the qubit, satisfying the commutation relation $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$. The bosonic operators $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the cavity photons. In Eq. (1), the constant γ is the spontaneous emission rate, $\mathrm{n_{th}} = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the mean number of thermal photons, k_B is the Boltzmann's constant, T is the temperature, and ω is the frequency. The superoperator $\mathcal{L}[\hat{x}, \hat{x}^\dagger]\hat{\rho} = \hat{x}\hat{\rho}\hat{x}^\dagger - (1/2)(\hat{x}^\dagger\hat{x}\hat{\rho} + \hat{\rho}\hat{x}^\dagger\hat{x})$ is the standard Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) dissipator [14].

2.1. Effective Dynamics

To derive the effective dynamics, we consider a weakly coupled atom-field system in the dispersive limit, in which the qubit and the cavity are far detuned compared to the coupling strength ($\lambda = \frac{g}{\Delta} \ll 1$). In this regime, the atom and the field do not exchange energy directly. To obtain the effective Hamiltonian and dissipator, we use the method of small unitary rotations [10, 15]. The following unitary operator $\hat{U} = \exp[\lambda(\hat{a}^\dagger\hat{\sigma}_- - \hat{a}\hat{\sigma}_+)]$ is used to transform $\hat{H}_{\text{int}} \to \hat{H}_{\text{eff}} = \hat{U}\hat{H}_{\text{int}}\hat{U}^\dagger$. Using the standard expansion $e^{\lambda A}Be^{-\lambda A} = B + \lambda[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + ...$ and keeping terms up to second order in λ, it is easy to show that the effective system Hamiltonian has the form

$$\hat{H}_{\text{eff}} = \Delta\hat{a}^\dagger\hat{a} - \frac{g^2}{\Delta}\left(\hat{a}^\dagger\hat{a}\hat{\sigma}_z + \frac{\hat{\sigma}_z}{2} + \frac{1}{2}\right). \tag{2}$$

Following the same procedure, one can show that the effective ME is given by

$$\begin{aligned}
\frac{d}{dt}\hat{\rho}_{\text{eff}}(t) = &-\imath\big[\hat{H}_{\text{eff}}, \hat{\rho}_{\text{eff}}\big] + \gamma\big(\mathrm{n_{th}}+1\big)\Bigg[\mathcal{L}\left[\hat{\sigma}_-, \hat{\sigma}_+\right]\hat{\rho}_{\text{eff}} + \frac{g}{\Delta}\mathcal{L}\left[\hat{a}\hat{\sigma}_z, \hat{\sigma}_+\right]\hat{\rho}_{\text{eff}} + \frac{g}{\Delta}\mathcal{L}\left[\hat{\sigma}_-, \hat{a}^\dagger\hat{\sigma}_z\right]\hat{\rho}_{\text{eff}} \\
&+ \frac{g^2}{\Delta^2}\mathcal{L}\left[\hat{a}\hat{\sigma}_z, \hat{a}^\dagger\hat{\sigma}_z\right]\hat{\rho}_{\text{eff}} - \frac{g^2}{\Delta^2}\mathcal{L}\left[\hat{\sigma}_-, \big(\hat{a}^\dagger\hat{a}+1\big)\hat{\sigma}_+ + 2\hat{a}^{\dagger 2}\hat{\sigma}_-\right]\hat{\rho}_{\text{eff}} \\
&- \frac{g^2}{\Delta^2}\mathcal{L}\left[\big(\hat{a}^\dagger\hat{a}+1\big)\hat{\sigma}_- + 2\hat{a}^2\hat{\sigma}_+, \hat{\sigma}_+\right]\hat{\rho}_{\text{eff}}\Bigg] + \gamma\mathrm{n_{th}}\Bigg[\mathcal{L}\left[\hat{\sigma}_+, \hat{\sigma}_-\right]\hat{\rho}_{\text{eff}} - \frac{g}{\Delta}\mathcal{L}\left[\hat{\sigma}_+, \hat{a}\hat{\sigma}_z\right]\hat{\rho}_{\text{eff}} \\
&- \frac{g}{\Delta}\mathcal{L}\left[\hat{a}^\dagger\hat{\sigma}_z, \hat{\sigma}_-\right]\hat{\rho}_{\text{eff}} + \frac{g^2}{\Delta^2}\mathcal{L}\left[\hat{a}^\dagger\hat{\sigma}_z, \hat{a}\hat{\sigma}_z\right]\hat{\rho}_{\text{eff}} - \frac{g^2}{\Delta^2}\mathcal{L}\left[\hat{\sigma}_+, \big(\hat{a}^\dagger\hat{a}+1\big)\hat{\sigma}_- + \hat{a}^2\hat{\sigma}_+\right]\hat{\rho}_{\text{eff}} \\
&- \frac{g^2}{\Delta^2}\mathcal{L}\left[\big(\hat{a}^\dagger\hat{a}+1\big)\hat{\sigma}_+ + \hat{a}^{\dagger 2}\hat{\sigma}_-, \hat{\sigma}_-\right]\hat{\rho}_{\text{eff}} - \frac{g^2}{\Delta^2}\mathcal{L}\left[\hat{\sigma}_+, \hat{\sigma}_-\right]\hat{\rho}_{\text{eff}}\Bigg].
\end{aligned} \tag{3}$$

In the next subsection, we transform the above equation to include classical laser driving.

2.2. External Driving

In order to include the effects of external driving in Eq. (3), we first define a unitary operator $\hat{U}_1 = \hat{U}\hat{U}_0\hat{U}^\dagger$ where $\hat{U}_0 = e^{-\imath\alpha\hat{\sigma}_x}$ and $\alpha = \epsilon\tau_{\text{int}}$. Here, ϵ is the driving amplitude, τ_{int} is the interaction time and $\hat{\sigma}_x$ is the Pauli matrix. A transformation to the rotating frame of the laser driving is made with a unitary operator $\hat{U}_1$, applied to the density matrix $\tilde{\rho} = \hat{U}_1\hat{\rho}_{\text{eff}}\hat{U}_1^\dagger$. The next step is to omit the counter-rotating terms (RWA) and write the ME in the Fock space representation using $\tilde{\rho} = \sum_k \tilde{\rho}_k \otimes |k\rangle\langle k|$. After some algebra one can show that the ME takes the following form

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}_k(t) = \imath\frac{g^2}{\Delta}\left[\tilde{J}_k, \tilde{\rho}_k\right] &+ \gamma(\mathrm{n_{th}}+1)\frac{g^2}{\Delta^2}(k+1)(\hat{\alpha}_{zy} - \sin^2\alpha\mathbb{1})\tilde{\rho}_{k+1}(\hat{\alpha}_{zy} - \sin^2\alpha\mathbb{1}) \\ &+ \gamma\mathrm{n_{th}}\frac{g^2}{\Delta^2}k(\hat{\alpha}_{zy} + \sin^2\alpha\mathbb{1})\tilde{\rho}_{k-1}(\hat{\alpha}_{zy} + \sin^2\alpha\mathbb{1}) + \gamma(k - \mathrm{n_{th}})\frac{g^2}{\Delta^2}(\hat{A}\tilde{\rho}_k + \tilde{\rho}_k\hat{A}) \\ &+ \gamma\mathrm{n_{th}}\frac{g^2}{\Delta^2}(k+1)\sin^4\alpha\tilde{\rho}_{k+1} + \gamma(\mathrm{n_{th}}+1)\frac{g^2}{\Delta^2}k\sin^4\alpha\tilde{\rho}_{k-1} - \gamma\frac{g^2}{\Delta^2}X'_k\tilde{\rho}_k. \end{aligned} \tag{4}$$

Here, the operator $\hat{\alpha}_{zy}$ is given by $\hat{\alpha}_{zy} = \cos 2\alpha\hat{\sigma}_z + \sin 2\alpha\hat{\sigma}_y$, where $\hat{\sigma}_z$ and $\hat{\sigma}_y$ are Pauli matrices. Also the operator $\hat{A}$ is given by $\hat{A} = \sin^2\alpha\ \hat{\alpha}_{zy}$, $X'_k = (2\mathrm{n_{th}}+1)\bar{X} + \mathrm{n_{th}}\mathbb{1}$, where $\bar{X} = \sin^4\alpha(k+1)\mathbb{1} + \sin^4\alpha k\mathbb{1} + k\mathbb{1}$, $\mathbb{1}$ is a 2 by 2 identity matrix and $\tilde{J}_k = \frac{3}{2}\sin^2\alpha k\mathbb{1} + (k+\frac{1}{2})\hat{\alpha}_{zy} + \frac{1}{2}\mathbb{1}$.

We are interested in the analytical expression of the observables. In order to achieve this, we transform Eq. (4) into a basis where its possible to solve it analytical using generating functions. By defining $\hat{\sigma}'_z = \cos 2\alpha\hat{\sigma}_z + \sin 2\alpha\hat{\sigma}_y$, $\hat{\sigma}'_x = \hat{\sigma}_x$ and $\hat{\sigma}'_y = -\sin 2\alpha\hat{\sigma}_z + \cos 2\alpha\hat{\sigma}_y$, it is straight forward to verify that $[\hat{\sigma}'_x, \hat{\sigma}'_y] = 2\imath\hat{\sigma}'_z$. Hence, one can derive the following transformation operator $\hat{U}_2 = \begin{pmatrix}\cos\alpha & \imath\sin\alpha \\ \imath\sin\alpha & \cos\alpha\end{pmatrix}$. Using the above operator $\hat{U}_2$, we can transform Eq. (4) to a new basis $\hat{\rho}_k = \hat{U}_2^\dagger\tilde{\rho}_k\hat{U}_2$ and write the transformed ME as follow

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_k(t) = \imath\frac{g^2}{\Delta}\left[\hat{J}_k, \hat{\rho}_k\right] &+ \gamma(\mathrm{n_{th}}+1)\frac{g^2}{\Delta^2}(k+1)(\hat{\sigma}_z - \sin^2\alpha\mathbb{1})\hat{\rho}_{k+1}(\hat{\sigma}_z - \sin^2\alpha\mathbb{1}) \\ &+ \gamma\mathrm{n_{th}}\frac{g^2}{\Delta^2}k(\hat{\sigma}_z + \sin^2\alpha\mathbb{1})\hat{\rho}_{k-1}(\hat{\sigma}_z + \sin^2\alpha\mathbb{1}) + \gamma\mathrm{n_{th}}\frac{g^2}{\Delta^2}(k+1)\sin^4\alpha\hat{\rho}_{k+1} \\ &+ \gamma(\mathrm{n_{th}}+1)\frac{g^2}{\Delta^2}k\sin^4\alpha\hat{\rho}_{k-1} - \gamma\frac{g^2}{\Delta^2}X'_k\hat{\rho}_k + \gamma(k - \mathrm{n_{th}})\frac{g^2}{\Delta^2}(\hat{\sigma}_z\hat{\rho}_k + \hat{\rho}_k\hat{\sigma}_z). \end{aligned} \tag{5}$$

The exact solution of the above equation (5) for the zero temperature case ($\mathrm{n_{th}} = 0$) will be outlined in the next section.

3. General Solution

In this section, we write Eq. (5) in the matrix elements form $\rho_k^{(i,j)}(t)$ $(i, j = 0, 1)$ and solve each system analytically using the matrix generating function given by

$$\rho_s^{(i,j)}(x,t) = \sum_{k=0}^{\infty} x^k \rho_k^{(i,j)}(t), \tag{6}$$

where $|x| \leq 1$. If $\rho_s^{(i,j)}$ is known, the transformed matrix elements can be found from

$$\rho_k^{(i,j)}(t) = \frac{1}{k!}\frac{\partial^k}{\partial x^k}\rho_s^{(i,j)}(x,t)\bigg|_{x=0}. \tag{7}$$

The generating function system of equations are solved using the method of characteristics with some arbitrary initial conditions. If we assume that at $t = 0$ the walker is localized at site p ($p \in \mathbb{Z}$) with $\hat{\rho}(0) = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \otimes |p\rangle\langle p|$, where $a + b = 1$, $(a, b) \in \mathbb{R}_{\geq 0}$ and $z \in \mathbb{C}$, it is easy to show that $\rho_s^{(0,0)}(x, 0) = ax^p$ and $\rho_s^{(1,1)}(x, 0) = bx^p$. Using the method of characteristics and the initial conditions, one can show that the analytic expression for the populations of the ground state and excited state (for $\mathrm{n_{th}} = 0$) are given by

$$\begin{aligned}\rho_k^{(0,0)}(t) = \frac{1}{k!}\sum_{q=0}^{k}\binom{k}{q} a\cos 2\alpha(\sin^4\alpha - \cos^4\alpha e^{-\alpha' t\cos 2\alpha})^q(1 - ae^{-\alpha' t\cos 2\alpha})^{k+p-2q}(\sin^4\alpha)^{k-q} \\ \times (\cos^4\alpha)^{p-q}\frac{(p+k-q)!}{(p-q)!}(-1)^{-q}(\cos^4\alpha - \sin^4\alpha e^{-\alpha' t\cos 2\alpha})^{-p-k+q-1}\theta(p-q), \qquad (8)\end{aligned}$$

and

$$\begin{aligned}\rho_k^{(1,1)}(t) = \frac{1}{k!}\sum_{q=0}^{k}\binom{k}{q} b(\cos 2\alpha - 2)(\sin^4\alpha - (\sin^4\alpha + 1)^2 e^{-\alpha'(2-\cos 2\alpha)t})^q(\sin^4\alpha)^{k-q} \\ \times (1 - ae^{-\alpha'(2-\cos 2\alpha)t})^{k-2q+p}\frac{(p+k-q)!}{(p-q)!}(\sin^2\alpha + 1)^{2p-2q}(-1)^{p+k-2q} \\ \times (\cos^4\alpha - 2 - \sin^4\alpha(1 - e^{-\alpha'(2-\cos 2\alpha)t}))^{-p-k+q-1}\theta(p-q), \qquad (9)\end{aligned}$$

where $\binom{k}{q}$ denotes the binomial coefficient, $\alpha' = \frac{\gamma g^2}{\Delta^2}$ and $\theta(l)$ is the Heaviside step function

$$\theta(l) = \begin{cases} 1, & l \geq 0 \\ 0, & l < 0. \end{cases}$$

Using Eqs. (8) and (9), we obtain the probability to find the walker at site k, $P_k(t) = \rho_k^{(0,0)} + \rho_k^{(1,1)}$. After sufficiently long period of time, $\sum_{k=0}^{\infty}\rho_k^{(0,0)}(t \to \infty) = a$, $\sum_{k=0}^{\infty}\rho_k^{(1,1)}(t \to \infty) = b$, assuming $\alpha \geq 0$, the system settles down into the initial condition, $\sum_{k=0}^{\infty} P_k(t) = 1$. The probability to find the walker at site k is shown in Fig. 1 for various parameters. The off-diagonal elements (the coherences) are complex conjugates of each other, $\rho_k^{(1,0)} = \left(\rho_k^{(0,1)}\right)^*$. Following the same steps with initial condition $\rho_s^{(0,1)}(x, 0) = zx^p$, the solution for the off-diagonal element is given by

$$\begin{aligned}\rho_k^{(0,1)}(t) = e^{-\gamma\tau} z\operatorname{cosec}(\tau)(\tan(\tau))^{-p}\left(\frac{\operatorname{cosec}^4\alpha}{2\gamma}\right)^{p-k}\frac{1}{k!}\sum_{q=0}^{k}\binom{k}{q}\frac{(p+q)!}{(p-k+q)!}\Big(\cot(\tau)\lambda - w\Big)^{-p-1-q} \\ \times \Big((w^2 + \lambda^2)\tan(\tau)\Big)^{p-k+q}\Big(\lambda + w\tan(\tau)\Big)^{k-q}\theta(p-k+q). \qquad (10)\end{aligned}$$

Here $w = 2\imath\Delta - 1 - 2\gamma\sin^4\alpha$, $\lambda = \sqrt{4\gamma^2\sin^4\alpha(\sin^4\alpha - 1) - w^2}$, $\tau = \frac{1}{2}t\varepsilon\lambda$ where $\varepsilon = \frac{g^2}{\Delta^2}$, and $z = x + \imath y$. Eqs. (8), (9) and (10) provide a complete solution for the matrix elements of the density operator Eq. (5). Hence, it is straight forward to transform these expressions ((8), (9) and (10)) to the original basis and construct the observables of interest (see Figs. 2, 3) using $W_k(t) = \mathrm{Tr}[\hat{\sigma}_W\tilde{\rho}_k(t)]$ i.e., $W_k \in (X_k, Y_k, Z_k)$ ($\hat{\sigma}_W$ is the corresponding Pauli matrix).

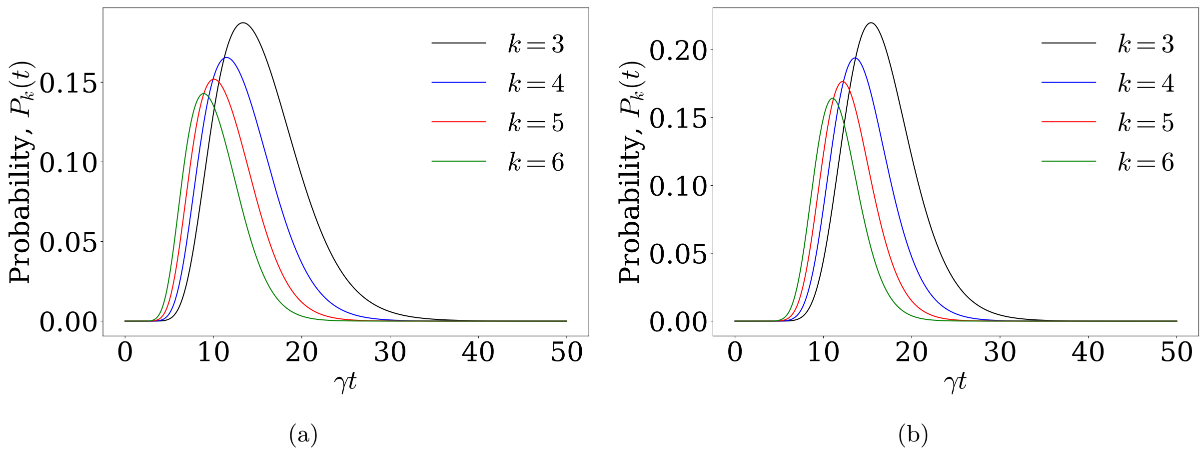


Figure 1: (Color online) The probability $P_k(t)$ of finding a walker at site k as a function of dimensionless time γt for different Fock states (stated in the legend). The initial sites are $p = 25$ (a) and $p = 35$ (b), with $\alpha = 0.3, 0.2$, respectively. Other parameters are set as $\gamma = 0.1$, $g = 0.4$ and $\Delta = 1$.

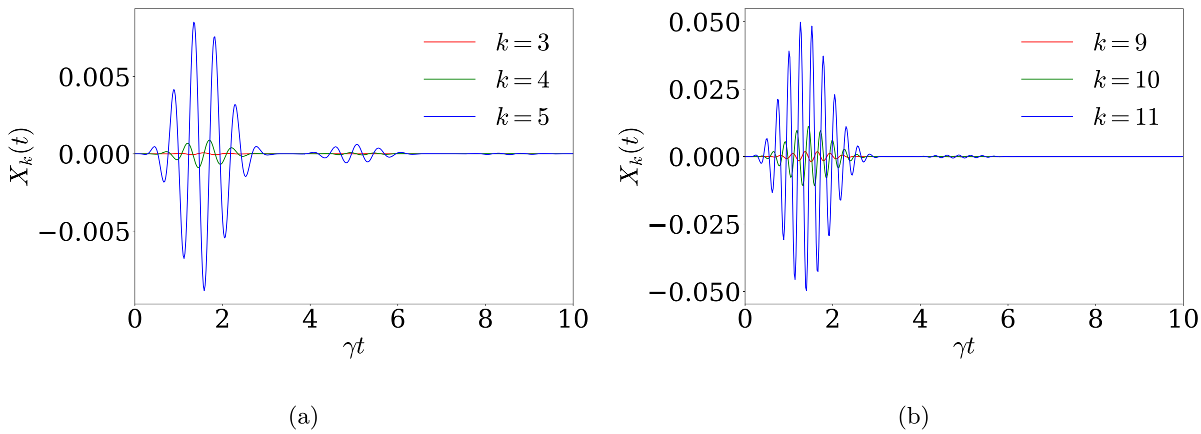
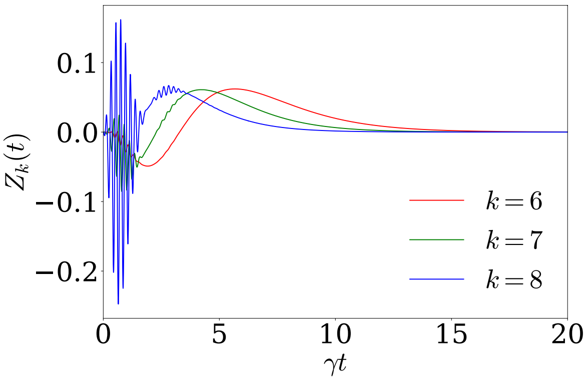


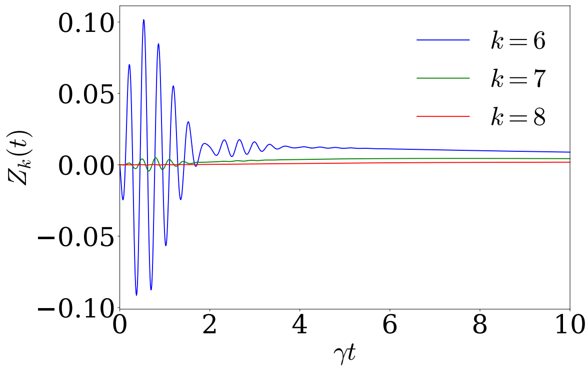
Figure 2: (Color online) The real part $X_k(t)$ of the coherences are shown as a function of the dimensionless time γt for different Fock states (stated in the legend). The initial sites are $p = 9$ (a) and $p = 15$ (b), with $\alpha = 0.4, 0.2$, respectively. Other parameters are $\gamma = 0.1$, $g = 0.3$ and $\Delta = 1$.

4. Conclusion

In this contribution, we have proposed a scheme for implementing OQWs in a cavity QED setup. We derived the generalized master equation for this OQW from the effective master equation. Further, we solved the generalized master equation using generating functions and constructed the analytic expressions for the populations for the zero temperature case. In order to verify our results, the dynamics of the observables are presented for various parameters and their behavior corresponds to the usual physical dynamics, which display collapses and revivals in the evolution of the system observables. Future work will show how Eq. (4) implements an OQW on $\mathbb{Z}^+$ at discrete driving time steps.



(a)



(b)

Figure 3: (Color online) The population inversion $Z_k(t)$ is shown as a function of dimensionless time γt for different sites k (stated in the legend). The initial sites are $p = 10$ (a) and $p = 5$ (b), with $\alpha = 0.4, 0.7$, respectively. Other parameters are $\gamma = 0.1$, $g = 0.4$ and $\Delta = 1$.

Acknowledgements

This work is based upon research supported by the South African Research Chair Initiative, Grant No. 64812 of the Department of Science and Innovation and the National Research Foundation (NRF) of the Republic of South Africa. AZ acknowledge support in part by the NRF of South Africa (Grant No. 129457). The Grant-holder (AZ) acknowledges that opinions, findings and conclusions or recommendations expressed in any publication generated by the NRF supported research is that of the author(s), and that the NRF accepts no liability whatsoever in this regard.

References

[1] H.-P. Breuer and F. Petruccione, 2002 *The Theory of Open Quantum Systems* (Oxford University Press, Oxford).

[2] C. W. Gardiner and H. Haken, 1991 *The Quantum Noise* (Springer, Berlin).

[3] Y. Aharonov, L. Davidovich, and N. Zagury, 1993 *Phys. Rev.* A **48**, 1687.

[4] J. Kempe, 2003 *Contemp. Phys.* **44**, 307.

[5] Attal et al., 2012 *Phys. Lett.* A **376**, 1545; S. Attal et al., 2012 *J. Stat. Phys.* **147**, 832; I. Sinayskiy and F. Petruccione, 2012 *Phys. Scr.* **T 151**, 014077.

[6] K. Kraus, 1983 *States, Effects and Operations: Fundamental Notions of Quantum Theory* (Springer-Verlag, Berlin).

[7] S. E. Venegas-Andraca, 2012 *Quant. Inf. Proc.* **11**, 1015.

[8] I. Sinayskiy and F. Petruccione, 2012 *Quant. Inf. Proc.* **11**, 1301.

[9] I. Sinayskiy and F. Petruccione, 2019 *Eur. Phys. J. Spec. Top.* **227**, 1869-1883.

[10] I. Sinayskiy and F. Petruccione, 2014 *Int. J. Quantum Inform.* **12**, 1461010.

[11] I. Sinayskiy and F. Petruccione, 2015 *Phys. Rev.* A **92**, 0321205; I. Sinayskiy and F. Petruccione, 2013 *Open Syst. Inf. Dyn.* **20**, 1340007.

[12] P. Filipowicz, J. Javanainen, and P. Meystre, 1986 *Phys. Rev.* A **34**, 3077.

[13] E. T. Jaynes and F. W. Cummings, 1963 *Proc. IEEE.* **51**, 89.

[14] G. Lindblad, 1976 *Commun. Math. Phys.* **48**, 119-130; V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, 1976 *J. Math. Phys.* **17**, 821.

[15] A. B. Klimov and S. M. Chumakov, 2009 *A Group-Theoretical Approach to Quantum Optics* (Wiley-VCH, Darmstadt); M. O. Scully and M. S. Zubairy, 1997 *Quantum Optics (Cambridge University Press, Cambridge).*