Matters of the $R_h = ct$ universe

Amare Abebe

Centre for Space Research, North-West University, Mahikeng 2745, South Africa
National Institute for Theoretical and Computational Sciences (NITheCS), South Africa

E-mail: `Amare.Abebe@nithecs.ac.za`

Abstract. Decades of astronomical observations have shown that the standard model of cosmology based on General Relativity - the closest we have to a standard theory of gravitation - does not adequately describe our universe without the *ad hoc* introduction of dark matter, dark energy to late-time cosmology and inflation to early-universe cosmology. This certainly has created dilemmas in the cosmology and the wider astronomy community, and several alternative models of cosmology and gravitation are being considered at the moment. Here I will give a brief overview of the cosmological dynamics of the $R_h = ct$ universe, a cosmological model with a vanishing active gravitational mass. We will show that the model presents, *inter alia*, a theoretical solution to the synchronicity problem in the framework of non-standard forms of matter and gravitation.

1. Introduction

Modern cosmology based on Einstein's General Relativity (GR) theory in a maximally symmetric spacetime, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [1, 2], describing a universe that is homogeneous (all regions of space look alike, no preferred positions) and isotropic (no preferred directions). The recent discovery of the accelerated rate of cosmic expansion has inspired a wave of new research into the nature of gravitational physics. Although it is not conclusively known what caused this recent cosmic acceleration, the prevailing argument is that dark energy caused it. Moreover, new alternatives to dark energy and/or generalisations to GR already abound [3, 4, 5], such as:

- Adding extra matter fields, e.g, scalar-tensor models, Modified Newtonian Dynamics [6].
- Adding extra dimensions, e.g., Kaluza-Klein [7], Gauss-Bonnet [8], braneworld models [9].
- Higher-order theories, e.g., Hořava-Lifshitz [10], $f(R)$ gravity [11] theories.

Melia's $R_h = ct$ spacetime model [12] is one such attempt to solve existing cosmological dilemmas that the concordance Λ-Cold-Dark-Matter (ΛCDM) cosmology has not resolved through a simple manipulation of the field equations in the existing GR framework. As we will show in the following section, this model requires the vanishing of the active gravitational mass (AGM). It is not possible to achieve a vanishing AGM in the standard description of GR if the universe is filled with standard forms of matter. This is because the energy density and the pressure of standard forms of matter are non-negative. Getting conditions for vanishing AGM in the standard framework are among the sticking points of the $R_h = ct$ universe, and the main objective of this article is to provide a mechanism to achieve this in the framework of modified theories of gravitation.

1.1. Matters of gravity

Relativistic physics treats gravity as a manifestation of spacetime curvature. There are three different geometrical representations of spacetime curvature [13] as shown in Fig. (1) leading to the three possible gravitational interpretations depicted in Fig. (2). In Fig. (1), we see the so-called geometric trinity: geometry manifesting itself as curvature - described by the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$, as torsion - described by the torsion tensor $T^{\alpha}{}_{\mu\nu}$, or as non-metricity - described by the eponymous tensor $Q^{\alpha}{}_{\mu\nu}$. Fig. (2) shows the three possible broad avenues to formulate a theory of gravitation: GR and its extensions in the curvature interpretation, the Teleparallel Equivalent of GR (TEGR) and its extensions in the torsion interpretation, and the Symmetric TEGR (STEGR) and its extensions in the non-metricity interpretation of geometry.

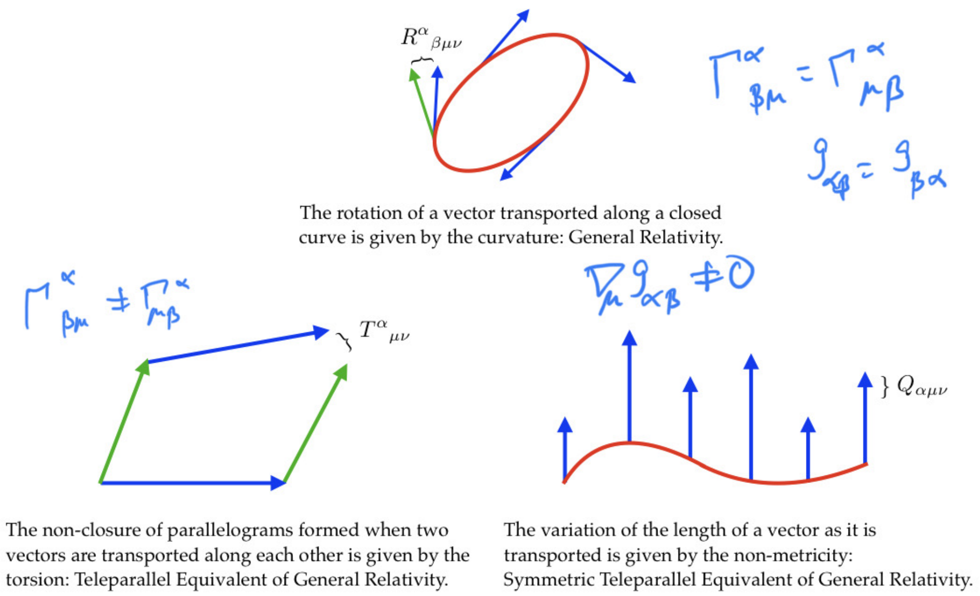

Figure 1. The geometrical meaning of curvature, torsion and non-metricity. Adapted from Beltran-Jimenez et al [13]. Here $R^{\alpha}{}_{\beta\mu\nu}$ is the Riemann curvature tensor, $\Gamma^{\alpha}{}_{\beta\mu}$ is the affine connection, and ∇_{μ} represents the covariant derivative operator. The failure of two parallel-transported vectors to form a parallelogram is given by the torsion tensor $T^{\alpha}{}_{\mu\nu}$ whereas the failure of the connection to be metric is given by the non-metricity tensor $Q^{\alpha}{}_{\mu\nu}$.

1.2. Standard cosmology

The standard (or "concordance") cosmological model is based on the GR description of gravitation the action of which is given by:

$$S_{GR} = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[R + 2 \left(L_m - \Lambda \right) \right] ,$$

where c and G are the speed of light and the Newtonian gravitational constant, g is the trace of the metric tensor $g_{\mu\nu}$, R is the Ricci scalar. The corresponding Einstein's field equations read [1]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \; , \tag{1}$$

with the first (geometric) term represented by the Einstein tensor, the right hand side of the equation representing the energy-momentum tensor of matter fluid forms, and Λ represents the cosmological constant. The other two descriptions of gravity, i.e., TEGR and STEGR [13] describe similar background expansion history provided $R = T = Q$ whereas their respective generalisations to $f(R)$, $f(T)$ and $f(Q)$ do not necessarily do so [14].

[1] From here onwards, unless explicit use is necessary, we will assume $8\pi G = 1 = c$.

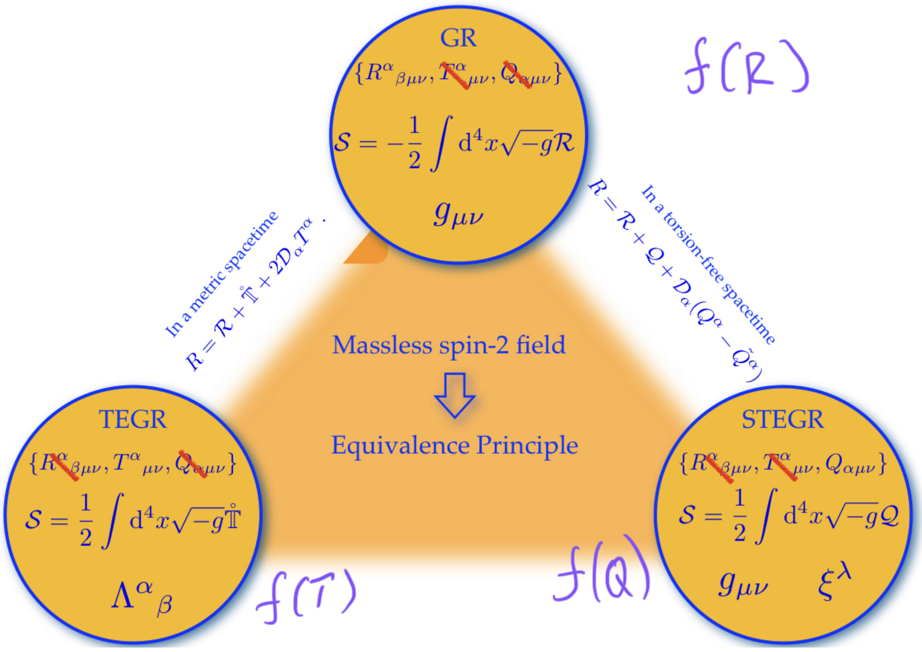

Figure 2. Three alternative gravitational descriptions defined in terms of the Ricci scalar R, torsion scalar T and non-metricity scalar Q, and their modified $f(R)$, $f(T)$ and $f(Q)$ generalisations. Adapted from Beltran-Jimenez et al [13]. In the GR case, all but the curvature tensor vanish, in TEGR only the torsion survives whereas in STEGER, the non-metricity survives while the curvature and torsion both vanish. Due to the vanishing curvature, there is a pure gauge field $\Lambda^\alpha{}_\beta$ associated with the connection in TEGR, and a tangent space coordinate ξ^α associated with STEGR.

In the absence of the cosmological constant in the above field equations (1), the basic evolution equation for the scale factor $a(t)$ of the universe is given by the Raychaudhuri equation

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p) ,$$

where ρ and p are the matter energy density and isotropic pressure, respectively, and an overdot represents differentiating with respect to (w.r.t) the cosmic time t. For ordinary matter, the "active gravitational mass"

$$\rho + 3p > 0 .$$

This means that ordinary matter will tend to cause the universe to decelerate, *i.e.,* $\ddot{a} < 0$. But astronomical observations show that cosmic expansion is accelerating in recent epoch, *i.e.,* $\ddot{a} > 0$. A positive cosmological constant causes an accelerated expansion, so Λ could be a quick fix provided that it dominates in the right-hand side of the following equation:

$$3\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p) + \Lambda .$$

Some serious problems associated with the cosmological constant, among them the eponymous *cosmological constant problem* [15] and the *coincidence problem* [16] challenge the choice of Λ as a dark energy candidate.

The Big Bang model is by far the most successful theory in predicting many cosmological features confirmed observationally expansion of space, origin of the cosmic microwave background, Big Bang nucleosynthesis, galactic evolution and distribution, and the formation of large-scale structures [4]. However, there are still, broadly speaking, two serious puzzles that remain unanswered in the standard Big Bang-based cosmology [17]:

- Early-universe problems: horizon, flatness, structure/smoothness/homogeneity, and magnetic-monopole problems. The suggested solution here is cosmic inflation, an early-epoch cosmic acceleration.
- Late-time (large-scale) problems: rotational curves of galaxies plus large-scale structure formation, and late-time cosmic acceleration. This is usually where the dark stuff comes in. The former aspect is generally thought to be solved by introducing dark matter, and the latter by dark energy (and often with the cosmological constant as the candidate for the dark energy).

One other puzzling, but not as widely explored, problem is the $R_h(t_0) = ct_0$ coincidence (often referred to as the synchronicity problem) [2]: the observed equality of our gravitational horizon $R_h(t_0)$ with the distance ct_0 light has travelled since the Big Bang in ΛCDMcosmology.

Some potential new frontiers to address the above shortcomings of the ΛCDMcosmology include:

- Inhomogeneous cosmological models, such as Lemaître-Tolman-Bondi (LTB) [18] and Szkeres [19] cosmologies.
- Anisotropic cosmological models, such as Bianchi cosmologies [20].
- Changing [fundamental] 'constants', such as evolving Λ and G cosmologies [21, 22, 23].
- Modification/generalization of GR, such as $f(R)$, $f(T)$ [24], and $f(Q)$ [25, 26] models of gravitation, just to mention a few.

In addition, the $R_h = ct$ model has recently joined the fray as a potential cosmological model to address, in particular, the synchronicity problem.

2. The $R_h = ct$ universe

First proposed by Melia, the controversial $R_h = ct$ model has the following characteristics [12, 27, 28]:

$$R_h = \frac{c}{H} = ct \;, \qquad a = t/t_0 \;, \qquad H = 1/t \Rightarrow Ht = 1 \quad \forall t \;. \tag{2}$$

Here R_h and $H \equiv \frac{\dot{a}}{a}$ are, respectively, the gravitational [horizon] radius and the Hubble parameter. The term c/H corresponds to the Hubble radius, also known as the Hubble horizon, and sets the scale for physically relevant causal relationships. The first part of Eq. (2) shows that the event horizon due to purely gravitational processes equals the Hubble horizon due purely to the cosmological expansion. This model follows from the imposition of a vanishing total AGM,

$$\rho + 3p = 0 \;, \tag{3}$$

in the gravitational theory. No cosmological constant Λ as the source of dark energy is needed in this model.

Among other things, this model resolves the so-called synchronicity problem: why is it true that today, the dimensionless age of the universe

$$H_0 t_0 \sim 1?$$

Here H_0 is the Hubble constant (*i.e.*, present-day value of the Hubble parameter). Problem: for standard forms of matter, $\rho + 3p > 0$. So how can one get a vanishing AGM? This is where modified models of gravitation come in as they naturally provide the extra "curvature", "torsion", etc... fluid terms that contribute to the total AGM. We will therefore assume from now on that the total fluid density and pressure terms are composed of pure matter forms denoted by the subscript $_m$ and the terms coming form the modifications in the form of curvature, torsion or non-metricity.

[2] Here t_0 denotes the age of the universe since the Big Bang.

2.1. Solutions from $f(R)$ gravity

$f(R)$ theories are a sub-class of *fourth-order* theories of gravitation [29, 3, 4, 30]:

$$S_{f(R)} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2L_m] \ ,$$

with corresponding generalized Einstein field equations:

$$f'G_{\mu\nu} = T^m_{\mu\nu} + \frac{1}{2}(f - Rf')g_{\mu\nu} + \nabla_\nu\nabla_\mu f' - g_{\mu\nu}\nabla_\gamma\nabla^\gamma f' \ .$$

Here and in the following, f , f' , etc. are shorthands for the arbitrary function $f(R)$ of the Ricci scalar R and its first, etc. derivatives w.r.t R. The sub/superscripts m and R stand for the standard matter and curvature components, respectively. The background curvature- fluid thermodynamics (introducing the volume expansion parameter $\Theta \equiv 3H$) can be represented as:

$$\rho_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R}\right] \ ,$$
$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R}\right] \ ,$$

whereas the total energy density and pressure terms can be given by

$$\rho \equiv \frac{\rho_m}{f'} + \rho_R \ , \quad p \equiv \frac{p_m}{f'} + p_R \ .$$

Imposing the vanishing AGM condition in $f(R)$ theories with energy and pressure terms as defined above, together with the flat-FLRW relation $R = 6H^2$ and matter solution for the $R_h = ct$ case:

$$\rho_m = \frac{\rho_{m0}}{a^{3(1+w)}} = 3H_0^2\Omega_{m0}\,(t_0/t)^{3(1+w)} = \alpha R^{3(1+w)/2} \ ,$$

where $\alpha \equiv 3t_0^{1+3w}\Omega_{m0}6^{-3(1+w)/2}$ and w is the equation of state parameter such that $p_m = w\rho_m$, we get the following second-order ordinary differential equation (ode) in f:

$$2R^2 f'' - f + 2\alpha R^{3(1+w)/2} = 0 \ . \tag{4}$$

Here we have used the fact that $\dot{R} = -\frac{2}{3}R\Theta = -\sqrt{\frac{2}{3}}R^{3/2}$. Moreover, we have introduced the notations ρ_{m0} and $\Omega_{\mathrm{m0}} \equiv \frac{\rho_{\mathrm{m0}}}{3H_0^2}$ as the present-day energy density of matter and its fractional value, respectively. Solving (4) gives

$$f(R) = c_1 R^{\frac{1+\sqrt{3}}{2}} + c_2 R^{\frac{1-\sqrt{3}}{2}} - \frac{4\alpha}{1 + 12w + 9w^2}R^{\frac{3(1+w)}{2}} \ , \tag{5}$$

with c_1 and c_2 integration constants. This same solution was obtained in [31]. This exact $f(R)$ solution was found [32] as a counter-example of the shear-free conjecture [33], i.e., describes a universe with simultaneous expansion and rotation. This then implies that the $R_h = ct$ universe is a simultaneously expanding and rotating spacetime solution! If one compares this result with the GR general condition [34], the $R_h = ct$ universe with $\rho + 3p = 0 \Rightarrow p = -\frac{\rho}{3} \Rightarrow c_s^2 = -1/3$, which is not an allowed solution for standard matter forms. This is an apparent contradiction and needs further investigation.

2.2. Solutions from $f(T)$ gravity

In the torsion (TEGR) interpretation of gravitation, the extended-TEGR action reads:

$$S_{f(T)} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(T) + 2L_m] \ ,$$

with corresponding modified field equations

$$f'G_{\mu\nu} = T^m_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(f - f'T) + f''S_{\mu\nu}{}^{\gamma}\nabla_\gamma T \ ,$$

where primes here denote differentiation w.r.t T and the last term denotes some "super-potential" related to the torsion. The background "torsion-fluid" and total thermodynamic quantities can be given by:

$$\rho_T = \frac{1}{f'}\left[\frac{1}{2}(Tf' - f)\right] \ ,$$
$$p_T = \frac{1}{f'}\left[\frac{1}{2}(f - Tf') + 2Hf''\dot{T}\right] \ ,$$

$$\rho \equiv \frac{\rho_m}{f'} + \rho_T = \frac{\rho_m}{f'} + \frac{1}{2f'}\left(Tf' - f\right) \ , \quad p \equiv \frac{p_m}{f'} + p_T = \frac{p_m}{f'} + \frac{1}{2f'}(f - Tf') + \frac{2Hf''\dot{T}}{f'} \ .$$

The vanishing AGM condition leads to the o.d.e

$$2T^2f'' - Tf' + f + (1 + 3w)\alpha(-T)^{\frac{3(1+w)}{2}} = 0 \ ,$$

the solution of which is given by

$$f(T) = c_3T + c_4\sqrt{T} - \frac{2\alpha}{2 + 3w}(-T)^{\frac{3(1+w)}{2}} \ , \qquad (6)$$

where c_3 and c_4 are integration constants. Hence any normal-matter-filled universe described by the $f(T)$-gravity needs to have the form of $f(T)$ Lagrangian density given by Eq. (6) if it is to satisfy Eq. (3) and thus describe the $R_h = ct$ universe. It is worth noting that particular solutions of this result can be shown to be a subset of the more general paradigmatic models of the extended teleparallel gravity solutions reconstructed in [35].

2.3. Solutions from $f(Q)$ gravity

In the STEGR interpretation of gravitation:

$$S_{f(Q)} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(Q) + 2L_m] \ .$$

The background non-metric and total-fluid thermodynamics are given by

$$\rho_Q = 3H^2f' - f/2 = \frac{1}{2f'}(Qf' - f) \ ,$$
$$p_Q = f/2 - \left(3H^2 + \frac{f''H\dot{Q}}{f'}\right)f' = \frac{1}{f'}\left[\frac{1}{2}(f - Qf') + 2Hf''\dot{Q}\right] \ ,$$
$$\rho \equiv \frac{\rho_m}{f'} + \rho_Q \ , \quad p \equiv \frac{p_m}{f'} + p_Q \ .$$

The vanishing AGM condition leads to the ode

$$2Q^2f'' - Qf' + f + (1 + 3w)\alpha Q^{\frac{3(1+w)}{2}} = 0 \ ,$$

the solution of which is given by

$$f(Q) = c_5 Q + c_6\sqrt{Q} - \frac{2\alpha}{2+3w} Q^{\frac{3(1+w)}{2}}\ , \tag{7}$$

c_5 and c_6 representing constants of integration, to be determined upon the imposition of initial conditions. This result shows that the background field equations describing a universe with normal matter and in which the underlying theory of $f(Q)$ gravitation has the Lagrangian density given by Eq. (7) satisfies the condition (3) and hence is a solution of the $R_h = ct$ universe.

3. Conclusion and outlook

The $R_h = ct$ universe is a possible cosmological alternative to ΛCDMin explaining certain issues such as the synchronicity problem. However, there are no sources of normal matter that satisfy the vanishing AGM $\rho + 3p = 0$ condition within the standard GR framework. We have demonstrated that this condition can be obtained from $f(R)$, $f(T)$ and $f(Q)$ gravity models provided that the gravity Lagrangians are as given by Eqs. (5), (6) and (7). In particular, we have shown that the $f(R)$ solution that satisfies vanishing AGM corresponds to the shear-free solution that allows a simultaneously expanding and rotating universe. This is a theoretical solution to the synchronicity problem in the framework of non-standard forms of matter and gravitation, and a lot needs to be done other cosmological aspects, such as perturbations, needs to further scrutinise these solutions for more cosmological viability.

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