Conditional Probability for One Particle Emission in a Scalar Field Theory

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Observation

For decades, inclusive radiative processes in QED and QCD have shown non-zero emission probabilities in **classically forbidden regions** of the phase space, naively violating energy conservation.

In particular, one may observe in the literature probability distributions with **support beyond** $x = 1$, where x is the fraction of the total energy carried off by the radiated particle. This is classically forbidden; an apparent violation of energy-momentum conservation!

This observation suggests that the matrix element $i\mathcal{M}$ is in some sense **disjoint from the limitations of the kinematics**, in that the matrix element is not necessarily zero beyond the physically permitted phase space defined through the kinematics.

This Motivates a Question

We would like to probe these observations of the large-x behaviour of radiative processes, and to answer the question:

Are **non-zero matrix elements** in classically forbidden regions physically reasonable, and what does this mean for the probability distribution?

Probing the Question in a Simpler Field Theory

We investigate this phenomenon by looking to a model free from the non-trivialities of gauge theories. We'd also like to isolate the physics of emission off a single leg, such that there is only one contributing Feynman diagram for the process. To this end, we **construct a scalar field theory**

$$\mathcal{L} := \mathcal{L}_{\text{free scalars}} + \mathcal{L}_{\text{interaction}}, \tag{1}$$

where $\mathcal{L}_{\text{free}} := \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C + \mathcal{L}_D$ such that

$$\mathcal{L}_A := \frac{1}{2}\partial_\mu \varphi_A \partial^\mu \varphi_A - \frac{1}{2} m_A^2 \varphi_A^2, \tag{2}$$

and similarly for $A \to B, C, D$. The quantities $\varphi = \varphi(x^\mu)$ are real scalar fields, and m_A denotes the mass of the particle field A, and similar for $A \to B, C, D$. For simplicity, we take $m_A = m_B = 0$. We take the interaction part of the Lagrangian to be

$$\mathcal{L}_{\text{interaction}} := \frac{1}{6} g_A \varphi_A^3 + g \varphi_A \varphi_B \varphi_C + \frac{1}{2} g_S \varphi_C \varphi_C \varphi_D, \tag{3}$$

where g, g_A and g_S are coupling constants.

Differential Cross-Section for One Particle Emission

In our scalar field theory, consider the **differential scattering cross-section** for the radiative process with Feynman diagram in Figure 1.

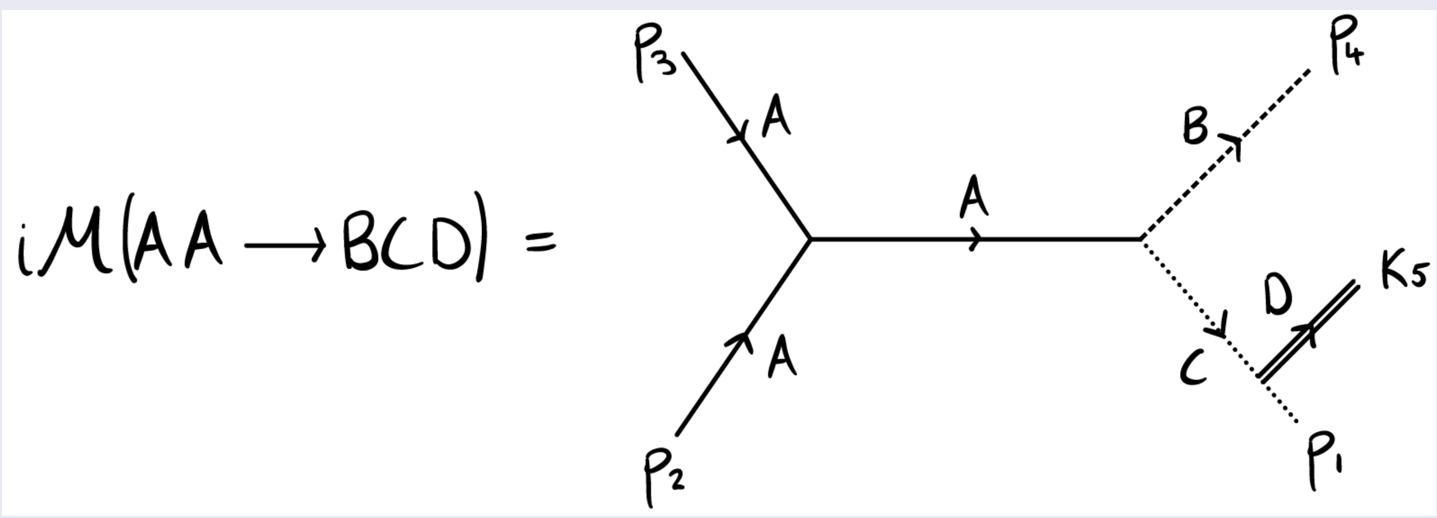

Figure 1: A Feynman diagram for one particle emission in our scalar field theory.

We have that

$$d\sigma^{2\to3} = \frac{1}{2s}\frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3k_5}{(2\pi)^3 2E_5}|\mathcal{M}(AA \to BCD)|^2 \times (2\pi)^4 \delta^{(4)}(p_2 + p_3 - p_1 - p_4 - k_5), \tag{4}$$

where $s := (p_2 + p_3)^2$. E_1, E_4 and E_5 are the energies of particles C, B and D respectively, and p_i is the momentum in the ith leg. The emitted particle D is taken to have momentum

$$k_5^\mu := (E_5, \sqrt{(x_5 E)^2 - \boldsymbol{k}_{5,\perp}^2 - m_D^2}, \boldsymbol{k}_{5,\perp})^\mu, \tag{5}$$

where $E_5 := x_5 E$ is the energy of the emitted particle, written in terms of the fraction x_5 of the total energy carried off by the emitted particle.

First Kinematic Constraint: Real Momenta

In order for our calculations to correspond to experimental observables, it is necessary that **all momenta are real numbers**, so we must enforce that the argument of the square root in k_5^z is positive. One may simply employ an overall Heaviside theta function to serve this purpose. This provides a lower bound on the energy for emission to be permitted!

Momentum Conservation

Next, one can readily perform the integral over the **momentum conserving Dirac delta function**,

$$\int d^3p_4 \frac{\delta^{(4)}(p_2 + p_3 - p_1 - p_4 - k_5)}{E_4(\boldsymbol{p}_4)} = \frac{\delta(2E - E_1 - E_4(\boldsymbol{p}_1, \boldsymbol{k}_5) - E_5)}{E_4(\boldsymbol{p}_1, \boldsymbol{k}_5)}, \tag{6}$$

where we used that $\boldsymbol{p}_2 + \boldsymbol{p}_3 = 0$ and $E_2 = E_3 = E$. The spatial integral over p_4 fixes $\boldsymbol{p}_4 = -(\boldsymbol{p}_1 + \boldsymbol{k}_5)$, and so $E_4(\boldsymbol{p}_4) \to E_4(\boldsymbol{p}_1, \boldsymbol{k}_5)$.

Energy Conservation

The integral over the **energy conserving Dirac delta function** fixes the permitted values of p_1^z such that the total energy is conserved throughout the process:

$$\int dp_1^z \delta(2E - E_1(\boldsymbol{p}_1, m_C) - E_4(\boldsymbol{p}_1, \boldsymbol{k}_5) - E_5) = \int dp_1^z \frac{\delta(p_1^z - p_{1,0+}^z) + \delta(p_1^z - p_{1,0-}^z)}{J(\boldsymbol{p}_1, \boldsymbol{k}_5, m_C, m_D)}, \tag{7}$$

where J is a Jacobian for the change of variables in the Dirac delta function, and we define $p_{1,0\pm}^z$ to be the two solutions of the equation

$$\left[((2 - x_5)E)^2 - E_1(\boldsymbol{p}_1, m_C)^2 - E_4(\boldsymbol{p}_1, \boldsymbol{k}_5)^2\right]^2 = 4E_1(\boldsymbol{p}_1, m_C)^2 E_4(\boldsymbol{p}_1, \boldsymbol{k}_5)^2, \tag{8}$$

which was obtained by rearranging the equation for the roots of the argument of the energy-conserving Dirac delta function in the first line in Eq. (7).

Second Kinematic Constraint: Excluding Unphysical Energies

One obtains another kinematic constraint, now from the energy-conserving Dirac delta function in Eq. (7); we must enforce that

$$((2 - x_5)E)^2 - E_1(\boldsymbol{p}_1, m_C)^2 - E_4(\boldsymbol{p}_1, \boldsymbol{k}_5)^2 > 0. \tag{9}$$

This is motivated by the form of Eq. (8); we may in principle end up with solutions in which the **quantity in square brackets is negative**. This is unphysical, but we would be wholly unaware of this configuration should it arise, owing to this quantity being squared in Eq. (8) and therefore unsensitive to an overall minus sign.

Yet More Constraints from the Kinematics

Since all energies are squared therein, Eq. (8) is a quadratic polynomial in p_1^z. One may collect powers of p_1^z in order to extract the coefficients a, b and c that solve the quadratic equation

$$p_{1,0\pm}^z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{10}$$

where $c + bp_{1,0\pm}^z + a(p_{1,0\pm}^z)^2 = 0$. Upon inspection of Eq. (10), one notes that for our momenta to be measurable, we must ensure that the **discriminant is positive**, so that we have $p_{1,0\pm}^z \in \mathbb{R}$. Furthermore, we choose $p_1^z > 0$, so we are forced to impose further constraints on the overall sign of the coefficients to ensure that we always take $p_{1,0\pm}^z > 0$. This leads to an additional six conditions that constrain our choice of roots $p_{1,0\pm}^z$ depending on the overall sign of the coefficients.

Conditional Probability Distribution

One finally has all the bits and pieces required to calculate the total cross-section for the one particle emission process in our scalar field theory, by numerically evaluating the remaining phase space integrals in Eq. (4). One can now compute the quantity

$$\frac{d\sigma^{2\to3}}{dx_5} / \sigma^{2\to2}, \tag{11}$$

which gives the **conditional probability distribution** for the emission process, as weighted by the two-to-two (no-emission) process, with total cross-section denoted $\sigma^{2\to2}$. We plot this quantity as a function of x_5 in Figure 2.

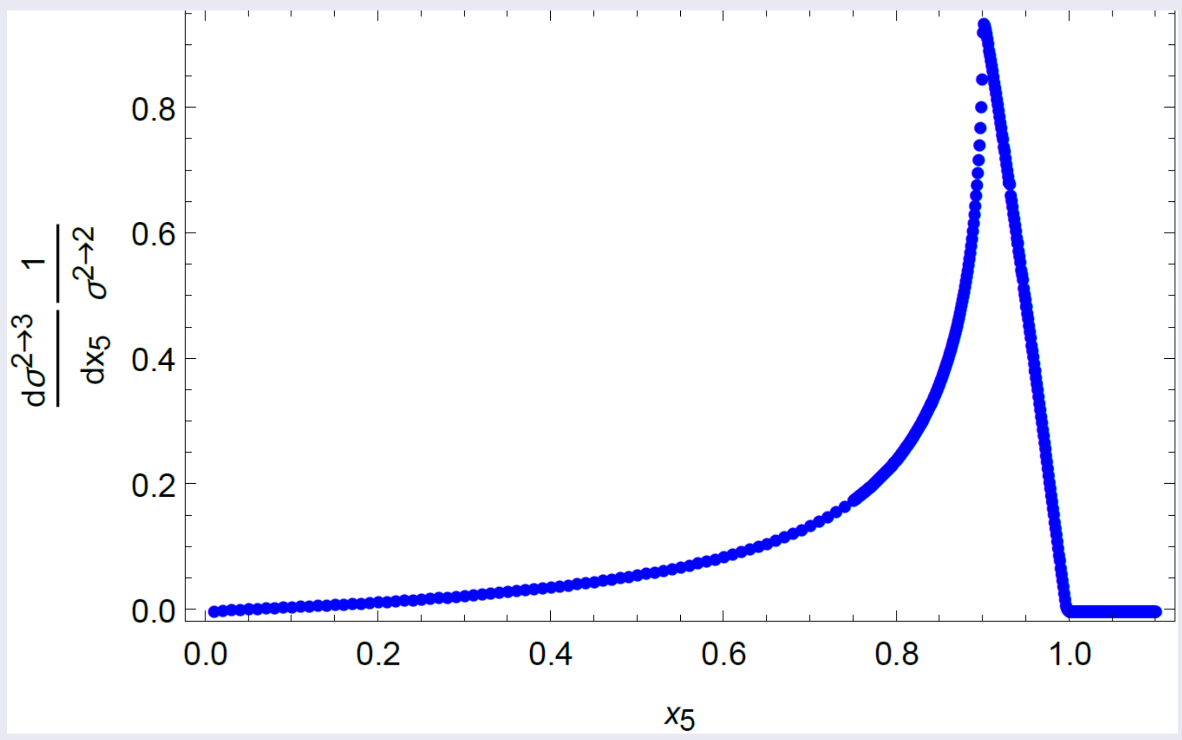

Figure 2: Conditional probability distribution for single particle emission in a scalar field theory defined by the Lagrangian in Eq. (1).

Discussion

We first note that our distribution has **enlargement at large** x_5, in contrast with particle emission within a gauge theory; there, one has enlargement at small x_5 due to the higher weight from the polarization vectors. This feature of our theory allows us to be very sensitive to the behaviour of the distribution at large x_5, near the kinematic cut-off at $x_5 = 1$.

The shape of the distribution suggests that the **matrix element is blowing up** as we approach the kinematic bound of $x_5 = 1$, but we also observe a **squeezing of the distribution** to zero as x_5 approaches 1. This suggests that the matrix element is not diverging so badly as to overcome the squeeze in the kinematics.

Conclusion

We have obtained results suggesting that indeed, the **matrix element need not necessarily vanish beyond the kinematic bounds**, but that any spillover of the probability distribution into this region is in fact suprious; correctly enforcing the kinematics leads to a smooth limit of the **probability going to zero at the kinematic bounds**, as expected by energy conservation.