Quasinormal modes in the large angular momentum limit: an inverse multipolar expansion analysis

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supervised by Prof. Alan Cornell

SAIP2021 | 27 July 2021
What are (black hole) quasinormal modes?

QNMs: an eigenvalue problem
2.1 The BH wave equation
2.2 Form of QNM potentials

Dolan-Ottewill multipolar expansion method
3.1 How it works
3.2 QNFs and QNM wavefunctions

Conclusions
Quasinormal mode: "ringdown"

Quasinormal mode: "ringdown"

Quasinormal mode and frequency

\[ \Psi(x^\mu) = \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r^{(d-2)/2}} e^{-i\omega t} Y_{\ell m}(\theta_i), \quad \omega_{sn\ell} = \omega_R - i\omega_I \]

- \( \mathbb{R}e\{\omega\} = \) physical oscillation frequency
- \( \mathbb{I}m\{\omega\} = \) damping \( \to \) dissipative, "quasi"
Quasinormal mode and frequency

\[ \Psi(x^\mu) = \sum_{\ell,m} \frac{\psi_{s\ell m}(r)}{r^{(d-2)/2}} e^{-i\omega t} Y_{\ell m}(\theta_i), \quad \omega_{s\ell m} = \omega_R - i\omega_I \]

- \( s \): spin of perturbing field
- \( m \): azimuthal number for spherical harmonic decomposition in \( \theta_i \)
- \( \ell \): angular/multipolar number for spherical harmonic decomposition in \( \theta_i \)
- \( n \): overtone number labels QNMs by a monotonically increasing \(|\text{Im}\{\omega\}|\)
What is the significance of these BH QNMs?

What theoretical gleanings can we extract?

- QNM frequencies are *independent* of exciting stimulus
  ⇒ encode characteristic information about the BH source
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- BH wave equations = window to QM description of BH?
  
  *Highly damped (asymptotic) QNFs related to area quantisation of BH event horizon?*

  = historical motivation for \(n \to \infty\)
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  retarded Green’s function in $d$-dimensional CFT
- BH wave equations = window to QM description of BH?
  Highly damped (asymptotic) QNFs related to area
  quantisation of BH event horizon?
  $= \text{historical motivation for } n \to \infty$
- $\ell \to \infty$: more than numerical aid?
The QNM eigenvalue problem

Black hole wave equation:

\[
\frac{d^2}{dx^2} \Phi(x) + [\omega^2 - V(r)] \Phi(x) = 0, \quad \frac{dr}{dx} = f(r)
\]
QNMs of integer spin:

\[ V_{eff}(r) = \frac{f(r)}{r^2} \left[ \ell(\ell + d - 3) + \frac{(d - 2)(d - 4)}{4} + \frac{P}{2r^{d-3} \mu} \right] \]

\[ f(r) = 1 - \frac{2\mu}{r^{d-3}} \]

<table>
<thead>
<tr>
<th>perturbation type</th>
<th>P</th>
</tr>
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<tbody>
<tr>
<td>scalar</td>
<td>((d - 2)^2)</td>
</tr>
<tr>
<td>electromagnetic: scalar*</td>
<td>(d(d - 4))</td>
</tr>
<tr>
<td>electromagnetic: vector</td>
<td>(-(d - 4)(3d - 8))</td>
</tr>
<tr>
<td>gravitational: vector</td>
<td>(-3(d - 2)^2)</td>
</tr>
<tr>
<td>gravitational: tensor*</td>
<td>((d - 2)^2)</td>
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\* \(d > 4\)
QNMs of half-integer spin:

\[ V_{1,2}(r) = \pm F(r) \frac{\partial}{\partial r} W + W^2 \]

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<th>perturbation type</th>
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<th>( W )</th>
</tr>
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<tr>
<td>spin-1/2</td>
<td>( f )</td>
<td>( \sqrt{f \kappa/r} )</td>
</tr>
<tr>
<td>spin-3/2: TT*</td>
<td>( f )</td>
<td>( \sqrt{f \kappa/r} )</td>
</tr>
<tr>
<td>spin-3/2: non-TT</td>
<td>( f )</td>
<td>( \sqrt{f \kappa/r} \times z_{\lambda=0} )</td>
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</tbody>
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\* \( d > 4 \)

\[
\kappa^2 = \ell + \frac{(d-2)^2}{4} \left( 1 - \frac{d-4}{d-2} \frac{2\mu}{r^{d-3}} \right), \quad \kappa = \ell + \frac{(d - 2)}{2}
\]

\[
z_{\lambda=0} = \frac{\kappa^2 - \frac{(d-2)^2}{4} \left( 1 - \frac{d-4}{d-2} \frac{2\mu}{r^{d-3}} \right)}{\kappa^2 - \frac{(d-2)^2}{4} \left( 1 - \frac{2\mu}{r^{d-3}} \right)}, \quad \kappa = \ell + \frac{(d - 2)}{2}
\]
The QNM eigenvalue problem

Black hole wave equation:

\[
\frac{d^2}{dx^2} \Phi(x) + [\omega^2 - V(r)] \Phi(x) = 0, \quad \frac{dr}{dx} = f(r)
\]

→ just a second-order ODE?
The QNM eigenvalue problem

Black hole wave equation:

\[
\frac{d^2}{dx^2} \Phi(x) + \left[ \omega^2 - V(r) \right] \Phi(x) = 0, \quad \frac{dr}{dx} = f(r)
\]

→ actually: QNM boundary conditions

purely ingoing: \( \Phi(x) \sim e^{i\omega x} \) \( x \to -\infty \) \( r \to r_H \) \( 1 \)
purely outgoing: \( \Phi(x) \sim e^{-i\omega x} \) \( x \to +\infty \) \( r \to +\infty \) \( 2 \)

Waves escape domain of study at the boundaries ⇒ dissipative
Dolan & Ottewill (2009)

A “new” computation method for BH QNMs through a novel ansatz based on null geodesics + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(x)}v(r) \, , \, \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

What are the major selling points?

- more efficient means of calculating detectable BH QNMs
- physically-motivated method
- easily extended to wavefunction computation
Components of the ansatz

\[ v(r) = \exp \left\{ \sum_{k=0}^{\infty} S_k(r)L^{-k} \right\} , \quad z(x) = \int_{x}^{\infty} \rho(r)dx = \int_{b}^{x} b_c k_c(r)dx \]

\[ k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2} \]
Components of the ansatz

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\[ r_c = \left. \frac{2f(r)}{\partial_r f(r)} \right|_{r=r_c}, \quad b_c = \left. \sqrt{\frac{r^2}{f(r)}} \right|_{r=r_c}, \quad k_c(r)^2 = \left. \frac{1}{b^2} - \frac{f(r)}{r^2} \right|_{r=r_c} \]
Components of the ansatz

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We generalise the consequent ODE

\[ f(r) \frac{d}{dr} \left( f(r) \frac{dv}{dr} \right) + 2i\omega \rho(r) \frac{dv}{dr} + \left[ i\omega f(r) \frac{d\rho}{dr} + (1 - \rho(r)^2) \omega^2 - V(r) \right] v(r) = 0 \]

We solve iteratively for \( \omega_k \) and \( S'_k(r) \) and sub into \( \omega \)
\[ r_c = 3, \ b_c = \sqrt{27} \ \Rightarrow \ \rho(r) = \left(1 - \frac{3}{r}\right)\sqrt{1 + \frac{6}{r}} \]
QNF expansions for the Schwarzschild BH

\[ r_c = 3, \ b_c = \sqrt{27} \quad \Rightarrow \quad \rho(r) = \left(1 - \frac{3}{r}\right) \sqrt{1 + \frac{6}{r}} \]

Sub \( v(r) = \exp\left\{\sum_{k=0}^{6} S_k(r)L^{-k}\right\} \), \( \omega = \sum_{k=-1}^{6} \omega_k L^{-k} \) into ODE

\[ L^2: \quad 27\omega_{-1}^2 - 1 = 0 \quad \Rightarrow \quad \omega_{-1} = \pm \frac{1}{\sqrt{27}} \]

\[ L^1: \quad 2i\omega_{-1} \left(1 + \frac{6}{r}\right)^{1/2} \left(1 - \frac{3}{r}\right) S_0' + \frac{54\omega_{-1}\omega_0}{r^2} + \frac{27i\omega}{r^3} \left(1 + \frac{6}{r}\right)^{-1/2} = 0 \]

\[ \Rightarrow \quad \omega_0 = -\frac{i}{2\sqrt{27}} \]

\[ \Rightarrow \quad S_0'(r) = \frac{\sqrt{27}}{r(r+6)(r-3)} \left[ \left(1 + \frac{6}{r}\right)^{1/2} - \frac{\sqrt{27}}{r} \right] \]

\[ \vdots \]
QNF expansions for the Schwarzschild BH

\[ r_c = 3 \, , \, b_c = \sqrt{27} \implies \rho(r) = \left(1 - \frac{3}{r}\right)\sqrt{1 + \frac{6}{r}} \]

<table>
<thead>
<tr>
<th>(s)</th>
<th>(b_c(r) \sum_{k=-1}^{6} \omega_k L^{-k})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(L - \frac{i}{2} + \frac{7}{216L} - \frac{137}{7776L^2}i + \frac{2615}{1259712L^3} + \frac{590983}{362797056L^4}i - \frac{42573661}{39182082048L^5} + \frac{11084613257}{8463329722368L^6}i)</td>
</tr>
<tr>
<td>1</td>
<td>(L - \frac{i}{2} - \frac{65}{216L} + \frac{295}{7776L^2}i - \frac{35617}{1259712L^3} + \frac{3374791}{362797056L^4}i - \frac{342889693}{39182082048L^5} + \frac{74076561065}{8463329722368L^6}i)</td>
</tr>
<tr>
<td>2</td>
<td>(L - \frac{i}{2} - \frac{281}{216L} + \frac{1591}{7776L^2}i - \frac{710185}{1259712L^3} + \frac{92347783}{362797056L^4}i - \frac{7827932509}{39182082048L^5} - \frac{481407154423}{8463329722368L^6}i)</td>
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The uniform potential

\[ \bar{L} - \frac{i}{2} - \frac{19}{108L} + \frac{295}{7776L^2}i + \frac{3853}{2519424L^3} - \frac{66089}{362797056L^4}i - \frac{165538573}{39182082048L^5} + \frac{54780211001}{8463329722368L^6}i\]

Perturbations of half-integer spin

| \(1/2\) | \(\bar{L} - \frac{i}{2} - \frac{11}{216L} - \frac{29}{7776L^2}i + \frac{1805}{1259712L^3} + \frac{27223}{362797056L^4}i + \frac{23015171}{39182082048L^5} - \frac{6431354863}{8463329722368L^6}i\) |
| \(3/2\) | \(\bar{L} - \frac{i}{2} - \frac{155}{216L} + \frac{835}{7776L^2}i - \frac{214627}{1259712L^3} + \frac{25750231}{362797056L^4}i - \frac{2525971453}{39182082048L^5} + \frac{292606736465}{8463329722368L^6}i\) |
Large multipolar limit: QNFs of a 4D Schwarz. BH

<table>
<thead>
<tr>
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<tr>
<td>10</td>
<td>$2.0213 - 0.0963i$</td>
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new!
Large multipolar limit: QNFs of a 4D Schwarz. BH

QNFs of integer spin → decrease

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QNFs of half-integer spin → increase

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Relationship between features of the QNFs in the large-\( \ell \) limit and the Lyapunov exponent (\( \Lambda \)) for spins \( s \in \{0, 1/2, 1, 3/2, 2\} \).

\[
\begin{array}{lcl}
\Re\{\Delta \omega\} & 0.1925 \pm 0.0001 \\
\Lambda & 0.19245 \\
|\Im\{\omega\}| & 0.0962 \pm 0.0001 \\
\Lambda/2 & 0.0962 \\
\end{array}
\]

Applicable beyond spin-2!

\[
\omega_{n,\ell \to \infty} = \Omega \left( \ell + \frac{1}{2} \right) - i \Lambda \left( n + \frac{1}{2} \right)
\]

\[
\Lambda = \frac{1}{\sqrt{27}} \approx 0.19245
\]

\( \Omega = \) orbital frequency of the \( r_{\text{orb}} = 3 \) photon sphere
Dolan & Ottewill (2009)

A “new” computation method for BH QNMs through a novel ansatz based on null geodesics + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(x)} v(r), \quad \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

What are the major selling points?

- more efficient means of calculating detectable BH QNMs
- physically-motivated method
- easily extended to wavefunction computation
Scalar QNM with $L=4$

$\Phi$

EM QNM with $L=4$

$\Phi$

Grav QNM with $L=4$

$\Phi$

Dirac QNM with $L=4$

$\Phi$

Rarita-Schwinger QNM with $L=4$

$\Phi$

- $\text{Re}(\Phi(r))$
- $\text{Im}(\Phi(r))$
Large multipolar limit: Dirac QNM wavefunction

Dirac QNM with $L = 4$

Graph showing the Dirac QNM wavefunction with $L = 4$. The graph plots the function $\Phi$ against $r$, with $r$ ranging from 4 to 12. The graph shows oscillatory behavior with peaks and troughs, typical of wave functions in quantum mechanics.
Large multipolar limit: Dirac QNM wavefunction

Dirac QNM with \( L = 10 \)

\[ \Phi \]

Large multipolar limit: Dirac QNM wavefunction
Large multipolar limit: Dirac QNM wavefunction

Dirac QNM with $L=20$
A highly economical method, from BH considerations

- Large-$\ell$ regime: physical insights, astrophysically relevant
  - $s$ less important as $\ell$ grows ($\mu = 1, n = 0$)
  - behaviours in $\Re\{\omega\}$, $\Im\{\omega\}$ for fixed $n, \ell$
A highly economical method, from BH considerations

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- DO validity in QNF computations:
  - Excellent agreement with literature + efficient
  - Possible to extend even further, beyond even RN + SdS!
  - $\rightarrow$ brewing: higher-$d$ BHs, AdS BHs...
Conclusions

A highly economical method, from BH considerations

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  - $s$ less important as $\ell$ grows ($\mu = 1$, $n = 0$)
  - behaviours in $\Re\{\omega\}$, $\Im\{\omega\}$ for fixed $n$, $\ell$

- DO validity in QNF computations:
  - Excellent agreement with literature + efficient
  - Possible to extend even further, beyond even RN + SdS!
    → brewing: higher-$d$ BHs, AdS BHs...

- As for the wavefunction $\Phi(r)$:
  - Straightforward computation but few physical insights
    → need to extend beyond radial profile
    e.g. QNM excitation coefficients via residues of GF poles
Thanks!
Spherically-symmetric, $d$-dimensional black hole

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{d-2}^2$$
Spherically-symmetric, $d$-dimensional black hole

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \]

Schwarzschild BH metric function:

\[ f(r) = 1 - \frac{2\mu}{r^{d-3}} \]

Event horizon ($f(r) = 0$): classically, no info may escape

\[ r_H = |2\mu|^{1/(d-3)} \]
The birth of black hole perturbation theory:

\[ g_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \quad \text{(linearised gravity)} \]
μ, θ, λ parametrise $M$, $Q$, $Λ$:

$$\mu = \frac{8\pi G_d}{(d-2) \Omega_{d-2}} M, \quad \theta^2 = \frac{8\pi G_d}{(d-2)(d-3)} Q^2, \quad \lambda = \frac{2\Lambda}{(d-1)(d-2)}$$

for the gravitational constant in $d$-dimensional space, $G_d$, and the area of a unit $(d-2)$-sphere, $\Omega_{d-2}$. 
Say we solve for the fundamental gravitational QNF of a 4D Schwarzschild BH. That is, $s = 2$, $\ell = 2$, $n = 0$. Recall: gravitational QNFs are isospectral. If we set $M = 1$,

$$\omega M \approx 0.3737 - 0.0890i$$

To convert from the Planck units we employ to kHz,

$$\omega \rightarrow \omega \times 2\pi(5.142 \text{ kHz}) \frac{M}{M_\odot}$$

For a 10 $M_\odot$ BH, we then have

$$\nu = (1.2074 - 0.2875i) \text{ kHz}$$
Gravitational perturbations: spin-2

\[
\frac{d^2}{dx^2} \Phi(x) + \left[ \omega^2 - V^{s=2}(r) \right] \Phi(x) = 0
\]

\[
\frac{dr}{dx} = f(r) = 1 - \frac{2M}{r}
\]

\( x \) maps \((rH, +\infty)\) to \((-\infty, +\infty)\)

Regge & Wheeler, 1957 (vector mode):

\[
V(r) = f(r) \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3} \right]
\]

Zerilli, 1970 (scalar mode):

\[
V(r) = f(r) \left[ \frac{2h^2(h + 1)r^3 + 6h^2Mr^2 + 18hM^2r + 18M^3}{r^3(hr + 3M)^2} \right],
\]

where \(2h = (\ell - 1)(\ell + 2)\)
Black hole wave equation:

\[
\frac{d^2}{dx^2}\Phi(x) + [\omega^2 - V(r)] \Phi(x) = 0
\]

\[
V_{\text{eff}}(r) = \frac{f(r)}{r^2} \left[ \ell(\ell + 1) + \frac{2M(1 - s^2)}{r} \right]
\]

\[
s = \begin{cases} 
0 , & \text{scalar} \quad \Rightarrow (1 - s^2) = 1 \\
1 , & \text{electromagnetic: vector mode} \quad \Rightarrow (1 - s^2) = 0 \\
2 , & \text{gravitational: vector mode} \quad \Rightarrow (1 - s^2) = -3 .
\end{cases}
\]
Gravitational: scalar
If we set $N = d - 2$,

$$V_S(r) = \frac{f(r)U(r)}{16r^2H(r)^2};$$

$$k_S^2 = \ell(\ell + d - 3), \quad \ell \in \mathbb{N}_0;$$

$$H(r) = k_S^2 - N + N(N + 1)\frac{\mu}{2r^{d-2}};$$

$$U(r) = -\lambda r^2 \left[ N^3(N + 2)(N + 1)^2 \left( \frac{\mu}{r^{d-3}} \right)^2 - 12N^2(N + 1)(N - 2)(k_S^2 - N) \left( \frac{\mu}{r^{d-3}} \right) + 4(N - 2)(N - 4)(k_S^2 - N)^2 \right]$$

$$+ N^4(N + 1)^2 \left( \frac{\mu}{r^{d-3}} \right)^3 + N(N + 1)(1 - f)^2 \left[ 4(2N^2 - 3N + 4)(k_S^2 - N) + N(N - 2)(N - 4)(N + 1) \right]$$

$$- 12N(k_S^2 - N) \left( \frac{\mu}{r^{d-3}} \right) \left[ (N - 4)(k_S^2 - N) + N(N + 1)(N - 2) \right] + 16(k_S^2 - N)^3 + 4N(N + 2)(k_S^2 - N)^2.$$
Test particle near BH ($\theta = \pi/2$):

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_{d-2}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left(-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2\right)$$

From the corresponding conjugate momenta,

$$p_t = f(r)\dot{t} \equiv E \Rightarrow \dot{t} = \frac{E}{f(r)}$$

$$p_\phi = r^2\dot{\phi} \equiv L \Rightarrow \dot{\phi} = \frac{L}{r^2}$$

$$p_r = f(r)^{-1}\dot{r}$$

$$\mathcal{H} = \left(p_t\dot{t} + p_\phi\dot{\phi} + p_r\dot{r} - \mathcal{L}\right)$$

$$\Rightarrow 2\mathcal{H} = E\dot{t} - L\dot{\phi} - f(r)^{-1}\dot{r}^2 = \delta_1$$

$\delta_1 = 1$: time-like geodesics

$\delta_1 = 0$: null geodesics
Introduce the definition: $\dot{r} \equiv V_r$

\[ \Rightarrow V_r = f(r) \left[ \frac{E^2}{f(r)} - \frac{L^2}{r^2} - \delta_1 \right] \]

For circular orbits: $V_r = V'_r = 0$

\[ \Rightarrow 0 = f(r) \left[ \frac{E^2}{f(r)} - \frac{L^2}{r^2} - 0 \right] \]

Thus, D & O define

\[ k_c(r) = \sqrt{\frac{E^2}{L^2} - \frac{f(r)}{r^2}} \quad \text{,} \quad b(r) = \frac{L}{E} \]

\[ k_c(r) \bigg|_{b_c, r_c} = 0 \]

\[ \Rightarrow b_c(r) = \sqrt{\frac{r^2}{f(r)}} \bigg|_{r_c} \quad \text{,} \quad r_c = \frac{2f(r_c)}{\partial_r f(r_c)} \]

DO origins
Efficient + economical method
⇒ highly consistent with 6th-order WKB

Observations:

- pronounced similarity from \( \ell = 10 \) (\( \mu = 1 \), \( n = 0 \))
- spin irrelevant for large angular dependence
- \( \Lambda \) suppresses, \( \theta \) enhances QNF growth
- confirms behaviour noted at lower \( \ell \) in literature:
  - Shu & Shen:
    - \( \Re \{ \omega \} \) ↓ with \( s \) for fixed \( n \) and \( \ell \)
    - \( \Re \{ \omega \} \) halfint ↑ with \( s \) for fixed \( n \) and \( \ell \)
- \( \Im \{ \omega \} \sim 0.0962 \) yields the correct value for Schwarz

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- higher orders (\( O(\ell - 6) > O(\ell - 2) \))
- some new int. + all new half-int. expressions
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