Non-Abelian Corrections for Radiation in QCD

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Abstract. We seek to compute the emission spectrum of soft and collinear gluon bremsstrahlung radiation associated with the hard scattering of a quark by a gluon in QCD for one, two, and three gluons. In QED, multiple photon emissions are independent, which is to say they are emitted according to a Poisson distribution. In QCD, the non-Abelian nature of the theory leads to interactions between the emitted gluons. Hence the emissions are not independent, and there are therefore corrections to the Poisson distribution of these radiated particles. We demonstrate how ideas from maximal helicity violating techniques will allow a calculation of these corrections and their potential relevance for heavy ion collision phenomenology.

1. Introduction

There is enormous interest in the scientific community in understanding better non-trivial, emergent, many-body dynamics. These emergent phenomena occur across vastly different materials, as varied as the constituent matter at the center of neutron stars to colonies of ants. Of the four fundamental forces—the gravitational, electromagnetic, weak, and strong—the strong force provides a unique laboratory to explore simultaneously both the theoretical and experimental implications for the emergent phenomena in a non-Abelian gauge theory.

One of the key observables in heavy ion collision phenomenology is jet quenching [1, 2]. In the collision of ultrarelativistic nuclei that provide experimental insight into the collective behavior of the strong force, rare high momentum particles are created that then propagate through the medium that is simultaneously created by the collision of nuclei. The interaction of these high momentum particles with the medium probes the emergent properties of the quark-gluon plasma (QGP); these interactions imprint themselves on the momentum distribution of the measured final state particles measured by detectors.

We therefore seek a theoretical understanding of these interactions in order to invert the measured observables to determine the properties of the quark-gluon plasma. If we assume that the coupling between these very high momentum particles and the medium is perturbatively calculable, then we expect that the dominant form of energy loss is radiative: the propagation of the high momentum particles is disturbed by the presence of the medium, which then leads to the emission of bremsstrahlung gluons. Calculations of the spectrum of these radiated gluons has been performed; see, e.g., [1, 2] for reviews. These derivations have so far involved only the single inclusive gluon emission spectrum. What one finds is that the expected number of emitted gluons is larger than one, more like three [3]. We expect the non-Abelian nature of

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QCD to lead to non-trivial correlations between these emitted gluons which are not currently captured in the single inclusive derivations performed so far.

Theoretical calculations with multiple gluon emission using standard perturbative QCD (pQCD) techniques is highly non-trivial; see, e.g., [4]. We therefore seek new methods for gaining insight into these multi-gluon emission processes. Amplitude and helicity techniques [5–8], which are often referred to as maximial helicity violation (MHV) techniques, appear to be one such promising tool.

2. A Model Problem

Performing pQCD calculations in the presence of a deconfined medium of quarks and gluons as is expected in the QGP produced in heavy ion collisions is a formidable problem. We thus seek to first apply amplitude techniques to the problem of $2 \to 2$ hard scattering of partons that also induces the emission of an additional n gluons. Significant progress was made in this problem using traditional pQCD techniques, especially for the case of collinear emission of 1 and 2 gluons [9]. Additional significant progress has been made applying MHV techniques to this process, with the successful determination of $g+g \to g+g+n\,g$ for n arbitrary when the emitted gluons are either fully symmetric or fully antisymmetric under interchange [10] and for $q+g \to q+g+n\,g$ where $n = 1$, 2, or 3 [11].

In this work we'll consider specifically the process of hard quark-gluon scattering with the emission of an additional gluon bremsstrahlung. One may find a number of reviews in the literature including, e.g., [6]. For the sake of clarity, we note some slight notational differences taken here. In this work we take

$$\varepsilon^{ab} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \varepsilon^{\dot{a}\dot{b}}; \qquad \varepsilon_{ab} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \varepsilon_{\dot{a}\dot{b}}. \tag{1}$$

and define

$$p_a = \varepsilon_{ab} p^b \qquad p_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} p^{\dot{b}} \tag{2}$$
$$p^a = \varepsilon^{ab} p_b \qquad p^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}} p_{\dot{b}}, \tag{3}$$

consistent with $p^{\dot{a}} = (p^a)^*$ and $p_{\dot{a}} = (p_a)^*$. Then

$$\langle pq \rangle \equiv p^a q_a = \varepsilon_{ab} p^a q^b = -\varepsilon_{ab} p^b q^a = -q^a p_a = -\langle qp \rangle \tag{4}$$
$$[pq] \equiv p_{\dot{a}} q^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}} p_{\dot{b}} q_{\dot{a}} = -\varepsilon^{\dot{a}\dot{b}} p_{\dot{a}} q_{\dot{b}} = -q_{\dot{a}} p^{\dot{a}} = -[qp]. \tag{5}$$

Since the angle and square brackets are antisymmetric, we have that $\langle pp \rangle = [pp] = 0$ for massless particles. Further, we have that

$$\langle pq \rangle^* = (\varepsilon_{ab} p^a q^b)^* = (\varepsilon_{ab})^* (p^a)^* (q^b)^* = \varepsilon_{\dot{a}\dot{b}} p^{\dot{a}} q^{\dot{b}} = p^{\dot{a}} q_{\dot{a}} = [qp]. \tag{6}$$

We then have the extremely important identity

$$\langle pq \rangle [qp] = p^a q_a q_{\dot{a}} p^{\dot{a}} = \varepsilon_{ab} p^a q^b \varepsilon_{\dot{a}\dot{b}} q^{\dot{b}} p^{\dot{a}} = \varepsilon_{ab} \varepsilon_{\dot{a}\dot{b}} p^\mu \bar{\sigma}_\mu^{\dot{a}a} q^\nu \bar{\sigma}_\nu^{\dot{b}b} = 2\eta_{\mu\nu} p^\mu q^\nu = 2p \cdot q \equiv s_{pq}, \tag{7}$$

where we've defined a generic Mandelstam variable $s_{pq} \equiv (p+q)^2$. Notice that the angle and square brackets are essentially square roots of dot products.

We work in $SU(N)$ and take $N \to 3$ at the end of the calculation. We follow the usual MHV conventions for the normalization of the Hermitian color generators T^a [6, 7], where a runs from 1 to $N^2 - 1$: $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ with $\mathrm{tr}(T^a T^b) = \delta^{ab}$. The introduction of this square root simplifies the representation of polarization vectors for external gluons. Let's define the color Casimir $\tilde{C}_F \equiv \frac{N^2-1}{N}$, which is twice the value of the usual Casimir, $\tilde{C}_F = 2C_F$, and for future use we define the second color Casimir $\tilde{C}_A \equiv N$.

For the purposes of applying MHV techniques, we define all lines as incoming in the amplitudes. In this way final-state momenta are assigned the negative of their physical momenta. Then momentum conservation for an n-point process takes the crossing symmetric form $\sum_{i=1}^{n} k_i = 0$. We also label the helicity as if the particle are incoming. For incoming particles this label is the physical helicity; for outgoing particles it is the opposite. In order to connect with the physical process of interest, $qg \to qg + ng$, we must apply crossing symmetry. Recall that when fermion lines are crossed, one must multiply by an overall minus sign to the result [12].

For the case of interest in this work, one finds that the full amplitude can be decomposed as [6, 7]

$$\begin{aligned}\mathcal{A}^{tree}_{\bar{q}g^n qg^m}(p_{\bar{q}}^{h_{\bar{q}}}, k_1^{h_1}, \ldots, k_n^{h_n}, p_q^{h_q}, \ell_1^{h_1}, \ldots, \ell_m^{h_m}) = \\ g_s^{n+m+2} \sum_{\sigma\in S_n, \tau\in S_m} (T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}} T^{a_{\tau(1)}} \cdots T^{a_{\tau(m)}})_{\alpha\beta} \times \\ A^{tree}_{\bar{q}g^n qg^m}(p_{\bar{q}}^{h_{\bar{q}}}, \sigma(k_1^{h_1}), \ldots, \sigma(k_n^{h_n}), p_q^{h_q}, \tau(\ell_1^{h_1}), \ldots, \tau(\ell_m^{h_m})). \quad (8)\end{aligned}$$

where the sum is performed over all possible permutations of the $n+m$ gluons and g_s is the usual strong coupling constant, $\alpha_s \equiv g_s^2/4\pi$. In this expression, $A^{tree}_{\bar{q}g^n qg^m}$ is known as a partial or color-stripped amplitude, which is gauge invariant. In the literature, various other names have been given to this gauge invariant component, such as color-ordered amplitude and dual amplitude [6, 7, 13]. Each partial amplitude in the sum corresponds to a particular color flow, which can be naively thought as the ordering in which the gluons are emitted. That the above decomposition is valid is a highly non-trivial result and depends on trading structure constants for color generators through the relationship $\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = \mathrm{tr}\left(T^a[T^b, T^c]\right)$ [6]. The color kinematic decomposition is especially powerful because the computation of the partial amplitude is significantly easier than the full amplitude. In particular, partial amplitudes have an extremely compact and simple form when they maximally violate helicity conservation. These MHV amplitudes have exactly two negative helicity external particles while all other external particles have positive helicity. This generalizes to the notion of N^{k}MHV amplitudes when $(k+2)$ external particles have negative helicity. The case where exactly two particles have positive helicity while all the other particles have negative helicity are called anti-MHV, ($\overline{\text{MHV}}$).

Using the spinor helicity formalism and the Britto–Cachazo–Feng–Witten (BCFW) on-shell recursion relation [14], one can show that a general MHV partial amplitude for a process which involves an arbitrary number of gluons and a quark-antiquark pair is given by [7]

$$A^{MHV}_{\bar{q}qg^n}(p_{\bar{q}}^-, p_q^+, 1^+, \ldots, k^-, \ldots, n^+) = \frac{\langle p_{\bar{q}} k\rangle^3 \langle p_q k\rangle}{\langle p_{\bar{q}} p_q\rangle \langle p_q 1\rangle \cdots \langle n p_{\bar{q}}\rangle}. \quad (9)$$

The partial amplitude for our MHV helicity configuration expressed in Eq. (9) is remarkably simple and only a function of the angle brackets. The $\overline{\text{MHV}}$ partial amplitude can be derived from Eq. (9) by swapping all helicities, which results in changing the angle brackets into square brackets and multiplying by an overall sign depending on the number of external legs. Thus $|A(\{h_i\})|^2 = |A(\{-h_i\})|^2$.

3. Results

Armed with Eq. (9) we may immediately compute the dominant soft and collinear emission of a single bremsstrahlung gluon radiation associated with $qg \to qg$ scattering. The relevant amplitude is

$$\mathcal{A}(1^-5^+2^+3^-4^+) \simeq g^3\langle 13\rangle^3\langle 23\rangle\Big(T^5T^3T^4\frac{1}{\langle 15\rangle\langle 52\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}+T^5T^4T^3\frac{1}{\langle 15\rangle\langle 52\rangle\langle 24\rangle\langle 43\rangle\langle 31\rangle}\Big), \tag{10}$$

where we have used the shorthand $T^i \equiv T^{a_i}$. We are able to drop the additional contributions to the amplitude $\mathcal{A}(1^-5^+2^+3^-4^+)$ from Eq. (8) for the following reason. We are interested in radiation soft and collinear to the outgoing quark, which is particle 1. Since angle brackets can be thought of as something akin to the square root of a dot product, $\langle 15\rangle$ is therefore very small when particle 5, the associated bremsstrahlung radiation for the hard scattering $qg \to qg$ process, is soft and collinear to the outgoing quark, particle 1. None of the other permutations of the three gluons will have a $\langle 15\rangle$ in the denominator and will therefore be small in comparison to the two terms included in Eq. (10).

Comparing to the $qg \to qg$ amplitude, one readily sees for radiation emitted soft and collinear to the outgoing quark that

$$\mathcal{A}(1^-5^+2^+3^-4^+) \simeq gT^5\frac{\langle 12\rangle}{\langle 15\rangle\langle 52\rangle}\mathcal{A}(1^-2^+3^-4^+). \tag{11}$$

The same logic follows for all other helicity configurations of the hard scattering subprocess; i.e. for helicities $\{h_i\}$ with $h_2 = +$ that are MHV,

$$\mathcal{A}(1^{h_1}5^+2^{h_2}3^{h_3}4^{h_4}) \simeq gT^5\frac{\langle 12\rangle}{\langle 15\rangle\langle 52\rangle}\mathcal{A}(1^{h_1}2^{h_2}3^{h_3}4^{h_4}). \tag{12}$$

When the emitted gluon has a negative helicity, the amplitude is severely suppressed by $\langle 15\rangle^3$ in the numerator when the diagram is MHV, i.e. when the other two gluons have positive helicity. However, when the emitted gluon has negative helicity and one of the hard gluons also has negative helicity, then the non-zero amplitudes (i.e. when the quark and anti-quark have opposite helicities) are $\overline{\text{MHV}}$. Thus for helicities $\{h_i\}$ with $h_2 = -$ that are $\overline{\text{MHV}}$,

$$\mathcal{A}(1^{h_1}5^-2^{h_2}3^{h_3}4^{h_4}) \simeq gT^5\frac{[12]}{[15][52]}\mathcal{A}(1^{h_1}2^{h_2}3^{h_3}4^{h_4}). \tag{13}$$

Upon squaring, these contributions will be the same as the squared contributions from the positive helicity emitted gluon MHV amplitudes; we will therefore simply multiply our final squared, summed result by 2 to account for these additional contributions.

We therefore have that

$$\langle|\mathcal{A}_5|^2\rangle = 2g^2\tilde{C}_F\frac{s_{12}}{s_{15}s_{52}}\langle|\mathcal{A}_4|^2\rangle, \tag{14}$$

$$\equiv |\mathcal{J}_g^{(1)}|^2\langle|\mathcal{A}_4|^2\rangle \tag{15}$$

where $\langle|\mathcal{A}_4|^2\rangle$ is given by the square of the $qg \to qg$ amplitude and we have defined the single gluon emission kernel $|\mathcal{J}_g^{(1)}|^2$ in the second line. Notice in the first line the critical overall factor of 2 due to the $\overline{\text{MHV}}$ contributions.

One may write the outgoing parton momenta in the usual way for high energy QCD processes and find that the emission spectrum derived above gives

$$\frac{dN_g^{(1)}}{d^3k} \simeq C_F\frac{\alpha_s}{\pi^2}\frac{1}{x}\frac{1}{k_\perp^2}, \tag{16}$$

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in exact agreement with the single gluon radiation spectrum of [15] associated with the scattering of a hard quark, in the limit of a massless quark.

Now consider the same process with two gluons soft/collinear with the outgoing fermion. We have that to leading order in kinematics, i.e. for terms enhanced by inverses of $\langle 15\rangle$, $\langle 16\rangle$, and/or $\langle 56\rangle$,

$$\begin{aligned}
&i\mathcal{M}(1_{\bar{q}}^- 2_q^+ 3^- 4^+ 5^+ 6^+) \\
&\quad= \sum_\sigma T^{\sigma(3)} \cdots T^{\sigma(6)} i\, M(1_{\bar{q}}^- 2_q^+ \sigma(3^+) \cdots \sigma(6^+)) \\
&\quad= i\, g^4 \langle 14\rangle^3 \langle 24\rangle \left[\frac{T^3T^4T^5T^6}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} + \frac{T^3T^4T^6T^5}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 46\rangle\langle 65\rangle\langle 51\rangle} + (3 \leftrightarrow 4) \right]
\end{aligned} \tag{17}$$

Just as in the previous case we have that

$$\begin{aligned}
\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} &= \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{[34]}{[34]} \frac{\langle 41\rangle}{\langle 45\rangle\langle 56\rangle\langle 61\rangle} \\
&\simeq -\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \frac{\langle 21\rangle}{\langle 25\rangle\langle 56\rangle\langle 61\rangle}.
\end{aligned} \tag{18}$$

The above equation will also hold for $5 \leftrightarrow 6$ and $3 \leftrightarrow 4$. Therefore we find that

$$\begin{aligned}
i\mathcal{M}(1_{\bar{q}}^- 2_q^+ 3^- 4^+ 5^+ 6^+) &\simeq i\, g^4 \frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle} \left[\frac{T^3T^4}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \quad \frac{T^5T^6}{\langle 25\rangle\langle 56\rangle\langle 61\rangle} + \frac{T^6T^5}{\langle 26\rangle\langle 65\rangle\langle 51\rangle} \Bigg) + (3 \leftrightarrow 4) \right] \\
&= i\mathcal{M}_4(1_{\bar{q}}^- 2_q^+ 3^- 4^+) \mathcal{S}_2(5,6),
\end{aligned} \tag{19}$$

where the two particle soft factor is

$$\mathcal{S}_2(5,6) \equiv \sum_\sigma T^{\sigma(5)} T^{\sigma(6)} S_2(\sigma(5), \sigma(6)), \tag{20}$$

where we'd defined the two particle soft factor partial amplitude

$$S_2(5,6) \equiv \frac{g^2 \langle 12\rangle}{\langle 25\rangle\langle 56\rangle\langle 61\rangle}. \tag{21}$$

One may straightforwardly brute force square the amplitude

$$\begin{aligned}
|\mathcal{S}_2(5,6)|^2 &= \left| \frac{g^2\langle 12\rangle}{\langle 56\rangle} \left(T^5T^5 \frac{1}{\langle 25\rangle\langle 61\rangle} - T^6T^5 \frac{1}{\langle 26\rangle\langle 51\rangle} \right) \right|^2 \\
&= \frac{g^4 s_{12}}{s_{56}} \left[\tilde{C}_F^2 \left(\frac{1}{s_{16}s_{25}} + \frac{1}{s_{26}s_{15}} \right) - \tilde{C}_F(\tilde{C}_F - \tilde{C}_A) \left(\frac{1}{\langle 25162]} + \text{c.c.} \right) \right]
\end{aligned} \tag{22}$$

We may simply this last expression to find

$$|\mathcal{S}_2(5,6)|^2 = g^4 \left[\tilde{C}_F^2 \frac{s_{12}^2}{s_{15}s_{25}s_{16}s_{26}} + \tilde{C}_F\tilde{C}_A \frac{s_{12}}{s_{56}} \frac{\text{tr}(\not{2}\not{5}\not{1}\not{6})}{s_{15}s_{25}s_{16}s_{26}} \right]. \tag{23}$$

One can clearly see that the first term is, up to an (important!) factor of 2, the "square" of the one gluon bremsstrahlung emission kernel:

$$\tilde{C}_F^2 \frac{s_{12}^2}{s_{15}s_{25}s_{16}s_{26}} = \frac{1}{2} |\mathcal{S}_1(5)|^2 |\mathcal{S}_1(6)|^2. \tag{24}$$

The second term is the non-Abelian correction to the "trivial" Poisson convolution that one finds in QED. (Notice that the non-Abelian correction goes to 0 as $\tilde{C}_A = N$ goes to 0.)

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4. Conclusions

The non-trivial challenge in using amplitude techniques on $qg \to qg + n\,g$ scattering for $n > 1$, as was shown here, is that next to maximal helicity violating diagrams must be included: including only the MHV (and $\overline{\text{MHV}}$) contributions as in [11] misses $1/2^{n-1}$ contributing helicity configurations. The calculation of these additional diagrams requires the use of the BCFW recursion relation. Although difficult, and perhaps somewhat tedious, these methods will produce results, and the number of diagrams that must be considered is significantly smaller than when using traditional pQCD techniques. There is therefore hope that we may be able to adapt these techniques to the physical processes that we believe are relevant in heavy ion collisions.

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