Perturbations in a Chaplygin gas Cosmology

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Abstract. In this paper we study the perturbations of a cosmic multi-fluid medium consisting of radiation, dust and a Chaplygin gas. To do so, we follow the $1+3$ covariant formalism and derive the evolution equations of the fluctuations in the energy density for each species of fluid in the multi-fluid system. The solutions to these coupled systems of equations are then computed in both short-wavelength and long-wavelength modes. Our preliminary results suggest that unlike most dark energy models that discourage large-scale structure formation due to the rapid cosmological expansion (which gives little time for fluctuations to coalesce), the Chaplygin-gas model supports the formation of cosmic structures. This is manifested in the solutions of the perturbation equations which show the growth of density fluctuations over time.

1. Introduction

In the last two decades, it has become apparent that the standard cosmological model (ΛCDM) fails to explain the accelerating expansion of the Universe [1], the existence of dark matter in galaxy clusters [2], the formation of the large-scale structure [3],the inherent inhomogeneity and anisotropy of the Universe on the small scales [4] and so forth. To solve the first two problems, scholars have proposed an exotic fluid, the so-called Chaplygin gas (CG), which acts as a cosmological fluid with an equation of state of the form $p = -\frac{A}{\rho^{\alpha}}$, where p and ρ are the pressure and energy density, and A and α are constants such that $A > 0$ and $0 < \alpha \leq 1$ [5]. The purpose of this fluid is to substitute the dark matter and dark-energy components of the Universe. This model acts as dark-matter in the early Universe and dark-energy in the late times of the cosmos. In a universe in which we assume the energy density to consist of matter and CG fluids, the conservation equation reads as

$$\dot{\rho}_t + 3\frac{\dot{a}}{a}(\rho_t + p_t) = 0, \tag{1}$$

where ρ_t is the energy density and p_t is the pressure for the total fluid[1], a is the cosmological scale factor and the subscript t stands for total (matter+CG) fluids[2]. From this equation, the energy

[1] When referring to matter, we mean either radiation or dust.
[2] w_m is the equation of state parameter for the matter fluid.

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density becomes $\rho_{tot}(z) = \left[A + B(1+z)^{3(1+\alpha)}\right]^{\frac{1}{1+\alpha}} + \rho_m(1+z)^{3(1+w_m)}$, where $B = e^{C(1+\alpha)}$ with C a constant of integration.

Whereas much has been said of the Chaplygin gas as an alternative to a unified dark matter and dark energy model that mimics the cosmic expansion history of a Friedmann-Lemaître-Robertson-Walker (FLRW) background, to our knowledge there is no work in the literature that considered the cosmological perturbations of this fluid model in the 1+3 covariant formalism [6–8]. Here we derive the density perturbation equations and present the solutions both analytically and numerically. From the results, we analyse cosmological implications as far as large-scale structure formation is concerned on both sub- and super-horizon scales [8].

The outline of the manuscript is as follows: in the following section we review the basic spatial gradient variables. In this section, we also derive the linear evolution equations, applying the scalar and harmonic decomposition techniques and obtain the wave-number dependent energy density fluctuations for both CG and matter fluids. In Sec. 3, we present the analytical and numerical solutions of density perturbations by considering radiation-CG, dust-CG and CG like fluids for both wave-length ranges. We then devote Sec. 4 to discussions of our results and the conclusions.

2. Perturbations

We define the spatial gradient variables for the energy density of the individual matter components ρ_m, the Chaplygin-gas energy density ρ_{ch}, the total energy density ρ_t and the volume expansion Θ respectively as

$$D^m_a = \frac{a}{\rho_m}\tilde{\nabla}_a\rho_m \ , \quad D^{ch}_a = \frac{a}{\rho_{ch}}\tilde{\nabla}_a\rho_{ch} \ , \qquad D^t_a = \frac{a}{\rho_t}\tilde{\nabla}_a\rho_t \ , \qquad Z_a = a\tilde{\nabla}_a\Theta \ .$$

For a perfect-fluid system, the following fluid equations hold:

$$\dot{\rho}_t = -\Theta(\rho_t + p_t) \ , \tag{2}$$
$$(\rho_t + p_t)\dot{u}_a + \tilde{\nabla}_a p_t \ , \tag{3}$$

from which one can conclude for the 4-acceleration

$$\dot{u}_a = \alpha\frac{\omega_{ch}\rho_{ch}D^{ch}_a}{a(\rho_t + p_t)} - \frac{w_m\rho_m D^m_a}{a(\rho_t + p_t)} \ , \tag{4}$$

where w_{ch} is the equation of state parameter of the CG fluid. Another key equation for a general fluid is the so-called Raychaudhuri equation and it can be expressed as

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\rho_t + 3p_t) - \tilde{\nabla}^a\dot{u}_a \ . \tag{5}$$

After defining the comoving gradients of the cosmological expansion and the comoving fractional density gradient of the matter components in a covariant and gauge-invariant way and taking the harmonically decomposed scalar parts, it can be shown that the scalar perturbations in the matter and CG energy densities evolve according to

$$\begin{aligned}\ddot{\Delta}^k_m = \quad & \Theta(\omega_m - \frac{2}{3})\dot{\Delta}_m + \Bigg\{(1+\omega_m)\Bigg[\frac{1}{2}(1+3\omega_m) + \frac{\Theta^2\omega_m}{3\rho_t(1+\omega_t)} \\ & + \frac{w(1+3\omega_t)}{2(1+\omega_t)} + \frac{k^2\omega_m}{a^2\rho_t}\Bigg]\rho_m + \Bigg[\frac{1}{3}\omega_m\Theta^2 - \frac{\omega_m}{2}(1+\omega_t)\rho_t\Bigg\}\Delta^k_m + (1+\omega_m) \\ & \Bigg[\frac{1}{2} - \frac{3\omega_{ch}}{2} - \frac{\Theta^2\omega_{ch}}{3\rho_t(1+\omega_t)} - \frac{\omega ch(1+3\omega_t)}{2(1+\omega_t)} + \frac{k^2\omega_{ch}}{a^2\rho_t}\Bigg]\rho_{ch}\ \Delta^k_{ch}\ , \qquad (6) \\ \ddot{\Delta}^k_{ch} = \quad & -\Theta\left(\omega_{ch} + \frac{2}{3}\right)\dot{\Delta}^k_{ch} + (1+\omega_{ch})\Bigg[\frac{1}{2}(1+3\omega) + \frac{\Theta^2\omega}{3\rho_t(1+\omega_t)} + \frac{\omega(1+3\omega_t)}{2(1+\omega_t)} \\ & + \frac{k^2\omega_m}{a^2\rho_t}\Bigg]\rho_m\Delta^k_m + \Bigg\{(1+\omega_{ch})\Bigg[\frac{1}{2} - \frac{3\omega_{ch}}{2} - \frac{\Theta^2\omega_{ch}}{3\rho_t(1+\omega_t)} - \frac{\omega_{ch}(1+3\omega_t)}{2(1+\omega_t)} \\ & + \frac{k^2\omega_{ch}}{a^2\rho_t}\Bigg]\rho_{ch} - \omega_{ch}\left(\frac{1}{3}\Theta^2 + \frac{1}{2}(1+3\omega_t)\rho_t\right) + \Theta^2\omega_{ch}\left[2(1+2\omega_{ch})\rho_{ch} + \frac{2}{3}\right]\Bigg\}\Delta^k_{ch}\ , \qquad (7)\end{aligned}$$

where

$$\Delta^k_m = a^2\frac{\nabla^2\rho_m}{\rho_m}\ , \qquad \Delta^k_{ch} = a^2\frac{\nabla^2\rho_{ch}}{\rho_m}\ , \qquad k = \frac{2\pi a}{\lambda}\ , \qquad (8)$$

k being the comoving wave-number and λ the wavelength of the perturbations. In redshift space, the corresponding equations can be recast as

$$\begin{aligned}\Delta''_m = \quad & \frac{1}{1+z}(\frac{1}{2} - 4\omega_m)\Delta'_m + \Bigg\{\frac{(1+\omega_m)}{(1+z)^2}\Bigg[\frac{1}{2}(1+3\omega_m) + \frac{\omega_m}{(\Omega_{ch}+\Omega_m)(1+\omega_t)} + \\ & \frac{\omega_m(1+3\omega_t)}{2(1+\omega_t)} + \frac{9\pi^2(1+\omega_m)^2\omega_m}{3\lambda^2(1+z)^{3(1+\omega_m)}(\Omega_{ch}+\Omega_m)}\Bigg]3\Omega_m + \Bigg[3\omega_m - \frac{3}{2}\omega_m(\Omega_{ch}+\Omega_m)(1+\omega_t)\Bigg\}\Delta^k_m \\ & + \frac{(1+\omega_m)}{(1+z)^2}\Bigg[\frac{1}{2} - \frac{3\alpha\omega_{ch}}{2} - \frac{\alpha\omega_{ch}}{(\Omega_{ch}+\Omega_m)(1+\omega_t)} - \frac{\alpha\omega_{ch}(1+3\omega_t)}{2(1+\omega_t)} + \\ & \frac{9\pi^2(1+\omega_m)^2\alpha\omega_{ch}}{3\lambda^2(1+z)^{3(1+\omega_m)}(\Omega_{ch}+\Omega_m)}\Bigg]3\Omega_{ch}\Delta^k_{ch}\ , \qquad (9) \\ \Delta''_{ch} = \quad & \frac{3\sqrt{4}-6}{\sqrt{4}(1+z)}\left(\omega_{ch}(1+\alpha) - \omega_{ch} + \frac{2}{3}\right)\Delta'_{ch} + \frac{(1+\omega_{ch})}{(1+z)^2}\Bigg[\frac{1}{2}(1+3\omega_m) + \frac{\omega_m}{3(\Omega_m+\Omega_{ch})(1+\omega_t)} \\ & + \frac{\omega_m(1+3\omega_t)}{2(1+\omega_t)} + \frac{9\pi^2(1+\omega_m)^2\omega_m}{3\lambda^2(1+z)^{3(1+\omega_m)}(\Omega_{ch}+\Omega_m)}\Bigg]3\Omega_m\Delta^k_m + \frac{1}{(1+z)^2}\Bigg\{(1+\omega_{ch})\Bigg[\frac{1}{2} - \frac{3\alpha\omega_{ch}}{2} \\ & - \frac{\alpha\omega_{ch}}{(\Omega_m+\Omega_{ch})(1+\omega_t)} - \frac{\alpha\omega_{ch}(1+3\omega_t)}{2(1+\omega_t)} + \frac{9\pi^2(1+\omega_m)^2\alpha\omega_{ch}}{\lambda^2(A+B(1+z)^{3(1+\alpha)})^{\frac{1}{1+\alpha}}(\Omega_{ch}+\Omega_m)}\Bigg]3\Omega_{ch} \\ & -\omega_{ch}\left(3 + \frac{3}{2}(1+3\omega_t)(\Omega_m+\Omega_{ch})\right) + 9\omega_{ch}\left[(1+\alpha)(1+2\omega_{ch})3\Omega_{ch} + \frac{2}{3}\right]\Bigg\}\Delta^k_{ch}\ , \qquad (10)\end{aligned}$$

respectively. In GR with normal forms of matter, one would obtain a closed second-order equation of the density fluctuations and the equations are generally easier to solve. But here we get a coupled system of second-order equations in the density fluctuations of both matter and CG, the solutions to which are are more complicated to compute analytically. As such, we consider short-wavelength ($k/aH \gg 1$) and long-wavelength ($k/aH \ll 1$) limits of the perturbations [8] and analyse large-scale structure implications of the resulting solutions.

3. Results and Discussion

For further discussion of the growth of the energy density fluctuations with redshift, we assume that the Universe has two major non-interacting fluids, namely radiation associated with CG and dust associated with CG[3], and CG fluids in the following subsections.

3.1. Radiation-CG dominated Universe

The equation of state parameter for radiation fluid is $w_r = 1/3$, and the equation of state parameter for CG reads

$$\omega_{ch} = -\frac{A}{A+B(1+z)^6}\ ,$$

The equation of state parameter of the total fluid is then given as

$$\omega_t = \frac{\rho_{0,r}(1+z)^4 - 3\frac{A}{\sqrt{A+B(1+z)^6}}}{3\rho_{0,r}(1+z)^4 + 3\sqrt{A+B(1+z)^6}}\ .$$

In the early universe, the redshift is large. If, in this case, we take B to be a small value, then $\omega_{ch} \approx -1$. The equation of state parameter for the total fluid consequently reads as

$$\omega_t \approx \frac{\rho_{0,r}(1+z)^4}{3\rho_{0,r}(1+z)^4 + 3\sqrt{B}(1+z)^3} = \frac{3H_0^2\Omega_{0,r}(1+z)}{9H_0^2\Omega_{0,r}(1+z) + 3\sqrt{B}}\ .$$

Now, if we again make the same assumptions of large redshift, z, and small values of B, then the total equation of state parameter becomes $\omega_t \approx 1/3 = \omega_r$.

If we assume radiation fluctuations in the background of the CG fluid, we have $\Delta_r \gg \Delta_{ch}$, i.e., $\Delta_{ch} \approx 0$, similar analysis is done in [8] for dust-radiation system. Then solutions of our leading equation, Eq. (9), for the short-wavelength range becomes

$$\begin{aligned}\Delta(z) = \quad & C_1(1+z)^{\frac{1}{12}}\text{BesselJ}\left(\frac{\Sigma}{24(\Omega_{ch}+\Omega_r)}, \frac{4\pi\sqrt{\Omega_r}}{3\lambda\sqrt{\Omega_{ch}+\Omega_r}(1+z)^2}\right) + \\ & C_2(1+z)^{\frac{1}{12}}\text{BesselY}\left(\frac{\Sigma}{24(\Omega_{ch}+\Omega_r)}, \frac{4\pi\sqrt{\Omega_r}}{3\lambda\sqrt{\Omega_{ch}+\Omega_r}(1+z)^2}\right)\ ,\end{aligned} \tag{11}$$

where $\Sigma = \sqrt{624\Omega_r^2 - 96\Omega_{ch}^2 + 528\Omega_{ch}\Omega_r + 145\Omega_{ch} + 289\Omega_r(\Omega_{ch}+\Omega_r)}$, C_1 and C_2 are integration constants and BesselJ and BesselY are the first- and second-order Bessel functions.

The solution of Eq. (9) for the long-wavelength range is given as

$$\Delta(z) = C_3(1+z)^{\Omega_{ch}+\Omega_r+\frac{\psi}{12(\Omega_{ch}+\Omega_r)}} + C_4(1+z)^{-(\Omega_{ch}+\Omega_r-\frac{\psi}{12(\Omega_{ch}+\Omega_r)})}\ , \tag{12}$$

where $\quad = \sqrt{432\Omega_{ch}^2\Omega_r + 1152\Omega_{ch}\Omega_r^2 - 96\Omega_{ch}^3 + 624\Omega_r^3 + 145\Omega_{ch}^2 + 289\Omega_r^2 + 434\Omega_{ch}\Omega_r}$. We assume initial conditions given as $\Delta_{in} \equiv \Delta^k(z_{in} = 1100) = 10^{-3}$ and $\Delta'_{in} \equiv \Delta'(z_{in} = 1100) = 0$ for every mode, k, to deal with the growth of matter fluctuations [8]. Therefore, we determine the integration constant C_i (i = 1,2,3,4,...) by imposing those initial conditions.

The numerical result of Eq. (11) is presented in Fig. 1. In this figure we observe the oscillatory motions of the density perturbations in the short-wavelength modes. The results of Eq. (12) are shown in Fig. 2 for the case of the long-wavelength modes. In this figure, we clearly observe the density fluctuations growing up through time.

[3] For all analysis, we consider the original CG model, $\alpha = 1$.

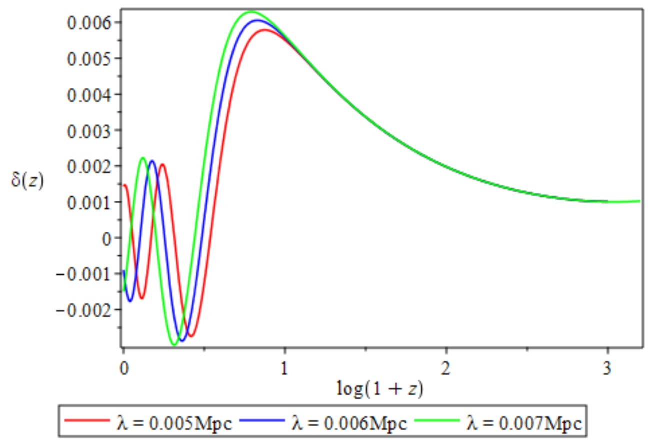

Figure 1. $\delta(z)$ versus z for short wavelengths for $\Omega_r = 1 - \Omega_{ch}$. Every colour represents a different wavelength.

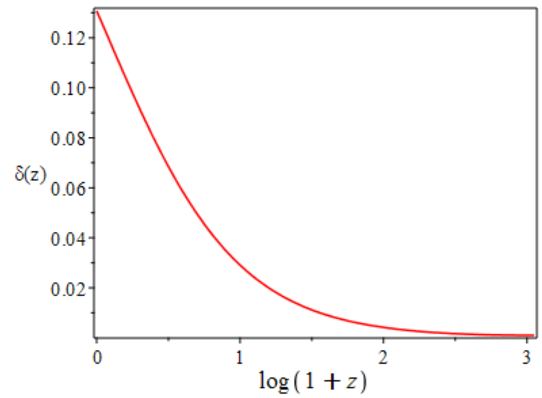

Figure 2. $\delta(z)$ versus z for long wavelengths for $\Omega_r = 1 - \Omega_{ch}$. We used $\Omega_r = 4.48 \times 10^{-5}$ [10].

3.2. Dust-CG dominated Universe

The equation of state parameter for dust fluid is given as $\omega_d = 0$, and the equation of state parameter of the total fluid then becomes $\omega_t = \frac{-\frac{A}{\sqrt{A+B(1+z)^6}}}{3\rho_{0,d}(1+z)^3+3\sqrt{A+B(1+z)^6}} \approx 0$, for large value of z. The solutions of our evolution equations for the density perturbations, which is given by Eq. (9), in a dust-CG system reads

$$\Delta(z) = C_5(1+z)^{\frac{3}{4}+\frac{1}{4}\sqrt{9+24\Omega_d}} + C_6(1+z)^{\frac{3}{4}-\frac{1}{4}\sqrt{9+24\Omega_d}} . \quad (13)$$

This solution is represented graphically in Fig. 3.

3.3. CG-dominated Universe

Considering the growth of the CG fluctuations as a background of matter fluids, we can let $\Delta_{ch} \gg \Delta_m$, causing $\Delta_m \approx 0$. The solution of our leading equation, Eq. (10), then reads as

$$\Delta(z) = C_7 \log(1+z)\sin(\Omega_r + \Omega_d + \Omega_{ch} + 5) + C_8 \log(1+z)\cos(\Omega_r + \Omega_d + \Omega_{ch} + 5) . \quad (14)$$

The numerical results are represented in Fig. 4 to depict the growth of CG density fluctuations in terms of redshift, and the role of CG fluid for the formation of large-scale structure.

What our current results show is that even at the level of the perturbations, the CG fluid offers an excellent alternative to the narrative of large-scale structures formation. This is because, contrary to what one would expect in a dark-energy-dominated universe where there would be less chance of large-scale structure formation due to the rapid cosmological expansion, we see the growth of the density perturbations with time.

4. Conclusions

In this work, we explored the solutions of cosmological perturbations in a multi-fluid cosmic medium where one of the fluids is a Chaplygin gas. We applied the $1+3$ covariant and gauge-invariant formalism to define the spatial gradient variables and applied scalar and harmonic decomposition methods to analyse the scalar perturbations of the different energy densities involved. We considered different systems such as radiation-CG, dust-CG and CG fluids in

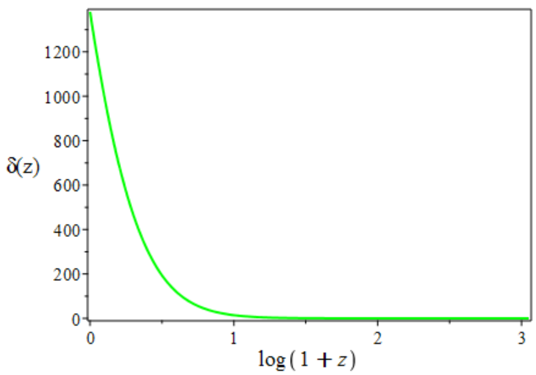

Figure 3. $\delta(z)$ versus z for dust and for $\Omega_d = 1 - \Omega_{ch}$. We used $\Omega_d = 0.32$ [11].

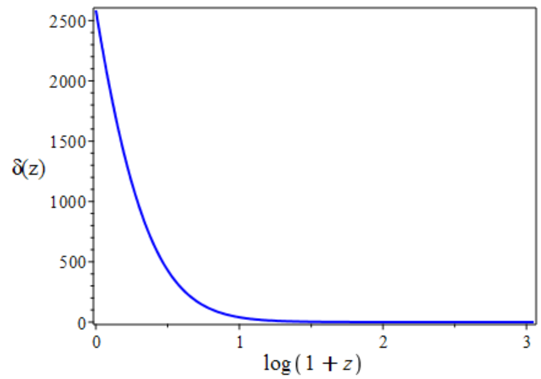

Figure 4. $\delta(z)$ versus z for CG, using $1 - \Omega_r - \Omega_d = \Omega_{ch}$.

both short- and long-wavelength modes to present the numerical and analytical solutions to the perturbation equations. Our results show that at least in the simplest CG model, the formation of large-scale structures is enhanced, rather than discouraged (as one would expect from dark energy fluid models), since all our preliminary calculations show the growth of density fluctuations with time.

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