ON THE EFFECTS OF DISSIPATION RANGE TURBULENCE ON THE PERPENDICULAR DIFFUSION COEFFICIENTS OF COSMIC RAY ELECTRONS

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Introduction
Why Perpendicular Diffusion Coefficients?
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Why Cosmic Ray Electrons?

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The aim is to investigate the effects of the omission of the dissipation range quantities on the perpendicular diffusion coefficients.
Turbulence Background
Turbulent Flow

Fluctuations are made up of eddies of different sizes. Each eddy tends to break up into smaller eddies, due to instabilities within the system. When the eddies reach a minimum size, their energy dissipates to the surrounding environment in the form of thermal energy.

Figure: [Davidson, 2004]
Turbulent Flow

Fluctuations are made up of eddies of different sizes. Each eddy tends to break up into smaller eddies, due to instabilities within the system. When the eddies reach a minimum size, their energy dissipates to the surrounding environment in the form of thermal energy.

Figure: [Davidson, 2004]

This process is called the energy cascade.
The amount of energy contained in the turbulent structures of the energy cascade is given by the turbulence power spectrum. This spectrum can be divided into ranges as follows:

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*Figure: [Nel, 2015]*

Due to physical considerations an inner range can also be included for the lowest wavenumbers.
We can express the heliospheric magnetic field as

\[ \mathbf{B}(x, y, z) = B_0 \hat{z} + \mathbf{b}(x, y, z) \]

where \( B_0 \) is the uniform background component and \( \mathbf{b} \) is the fluctuating component.

Different models can be used for \( \mathbf{b} \), namely

- **slab model**- assumes fluctuations are propagating in the \( z \)-direction
- **2D model**- assumes fluctuations are propagating in the \( (x,y) \)-plane
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A composite turbulence model consisting of 20% slab and 80% 2D turbulence can be used to approximate the heliospheric magnetic field at Earth [Bieber et al., 1996].
CALCULATION OF THE DIFFUSION COEFFICIENTS
The Random Ballistic Decorrelation interpretation of Non-linear Guiding Centre Theory [Ruffolo et al., 2012] presents a linear equation for $\kappa_\perp$ that can be solved analytically.

\[
\kappa_\perp = a^2 v^2 \frac{3}{2} B^2 0 \sqrt{\pi} 2 \int \epsilon^2 D(k_\perp) \sqrt{\sum_i k^2 i \langle v^2 \rangle} \times \text{erfc} \left[ v^2 3 \kappa_\parallel + \gamma (k_\perp) 2 \sqrt{\sum_i k^2 i \langle v^2 \rangle} \right] dk_\perp
\]

Note that:

\[
\kappa = v \lambda^3
\]

Assumptions:
- Axisymmetry
- Transverse fluctuations

Furthermore, we assume magnetostatic turbulence so $\gamma (k_\perp) = 0$. 

[-0x-0]
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The Perpendicular Diffusion Coefficient:

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \sqrt{\frac{\pi}{2}} \int \frac{e_{2D}(k_\perp)}{\sqrt{\sum_i k_i^2 \langle v_i \rangle^2}} \times \text{erfc} \left[ \frac{v^2}{3\kappa_\parallel} + \gamma(k_\perp) \right] \frac{\sqrt{\sum_i k_i^2 \langle v_i \rangle^2}}{2} \, dk_\perp$$

Note that:

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Note that: $\kappa = \frac{v \lambda}{3}$

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- Axisymmetry
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Furthermore, we assume magnetostatic turbulence s.t. $\gamma(k_\perp) = 0$
The Omnidirectional Power Spectrum

\[ \varepsilon_{2D}(k_\perp) = \frac{C_0 \lambda_{2D} \delta B_{2D}^2}{2\pi k_\perp} \begin{cases} \left( \frac{\lambda_{2D}}{\lambda_{out}} \right)^{-1} (\lambda_{out} k_\perp)^{-p}, & |k_\perp| < \frac{1}{\lambda_{out}} \\ (\lambda_{2D} k_\perp)^{-1}, & \frac{1}{\lambda_{out}} \leq |k_\perp| < \frac{1}{\lambda_{2D}} \\ (\lambda_{2D} k_\perp)^{-s}, & |k_\perp| \geq \frac{1}{\lambda_{2D}} \end{cases} \]

where \( p = 3 \) and \( s = \frac{5}{3} \). \( \delta B_{2D}^2 \) is the total magnetic variance.

The normalisation constant \( C_0 \) can be determined by setting:

\[ \int_0^\infty 2\pi k_\perp \varepsilon_{2D}(k_\perp) dk_\perp = \delta B_{2D}^2 \]

This yields:

\[ C_0 = \frac{(p + 1)(s - 1)}{\rho + s + (p + 1)(s - 1) \log \left( \frac{\lambda_{out}}{\lambda_{2D}} \right)} \]
"THE OMNIDIRECTIONAL POWER SPECTRUM

\[ \varepsilon_{2D}(k_{\perp}) = \frac{C_0 \lambda_{2D} \delta B_{2D}^2}{2\pi k_{\perp}} \begin{cases} 
(\frac{\lambda_{2D}}{\lambda_{out}})^{-1} (\lambda_{out} k_{\perp})^{-p} & , \quad |k_{\perp}| < \frac{1}{\lambda_{out}} \\
(\lambda_{2D} k_{\perp})^{-1} & , \quad \frac{1}{\lambda_{out}} \leq |k_{\perp}| < \frac{1}{\lambda_{2D}} \\
(\lambda_{2D} k_{\perp})^{-s} & , \quad |k_{\perp}| \geq \frac{1}{\lambda_{2D}}
\end{cases} \]

![Graph showing omnidirectional power spectrum with wavenumber on the x-axis and omnidirectional power spectrum on the y-axis, with a peak at a wavenumber of approximately 100 AU\(^{-1}\).]
The New Omnidirectional Power Spectrum

\[ \varepsilon_{2D}(k_\perp) = \frac{C_0 \lambda_{2D} \delta B_{2D}^2}{2\pi k_\perp} \begin{cases} 
(\frac{\lambda_{2D}}{\lambda_{out}})^{-1} (\lambda_{out} k_\perp)^{-p} & , \quad |k_\perp| < \frac{1}{\lambda_{out}} \\
(\lambda_{2D} k_\perp)^{-1} & , \quad \frac{1}{\lambda_{out}} \leq |k_\perp| < \frac{1}{\lambda_{2D}} \\
(\lambda_{2D} k_\perp)^{-s} & , \quad \frac{1}{\lambda_{2D}} \leq |k_\perp| < \frac{1}{\lambda_D} \\
(\frac{\lambda_D}{\lambda_{2D}})^s (\lambda_{D} k_\perp)^{-q} & , \quad |k_\perp| \geq \frac{1}{\lambda_D}
\end{cases} \]

where \( p = 3 \), \( s = \frac{5}{3} \) and \( q = 3 \).

Normalisation yields:

\[
C_0 = \left[ \frac{\lambda_{2D}^2}{(p+1)\lambda_{out}^2} + \frac{\lambda_{2D}}{(q-1)\lambda_D} + \frac{\lambda_D - \lambda_{2D}}{(s-1)\lambda_D} + \log \left( \frac{\lambda_{out}}{\lambda_{2D}} \right) \right]^{-1}
\]
The New Omnidirectional Power Spectrum

\[ \varepsilon_{2D}(k_\perp) = \frac{C_0 \lambda_{2D} \delta B_{2D}^2}{2\pi k_\perp} \left\{ \begin{array}{ll}
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(\lambda_{2D} k_\perp)^{-1} & , \frac{1}{\lambda_{out}} \leq |k_\perp| < \frac{1}{\lambda_{2D}} \\
(\lambda_{2D} k_\perp)^{-s} & , 1 \lambda_{2D} \leq |k_\perp| < 1 \lambda_D \\
(\frac{\lambda_D}{\lambda_{2D}})^{s}(\lambda_D k_\perp)^{-q} & , |k_\perp| \geq \frac{1}{\lambda_D}
\end{array} \right. \]
The equations for the parallel mean free path were derived from two different models, namely:

**The Damping Model**

\[
\lambda || = \frac{3s}{\sqrt{\pi(s-1)}} \frac{R^2}{k_{min}} \left( \frac{B_0}{\delta B_{slab}} \right)^2 \left( \frac{1}{4\sqrt{\pi}} + 2 F_1(1, \frac{1}{p-1}, \frac{p}{p-1}; -\frac{\pi a}{f_1} Q^{p-2}) \frac{\sqrt{\pi a}}{f_1 R^s Q^{p-s}} + \frac{2}{\sqrt{\pi(2-s)(4-s)}} \frac{1}{R^s} \right)
\]

and

**The Random Sweeping Model**

\[
\lambda || = \frac{3s}{\sqrt{\pi(s-1)}} \frac{R^2}{k_{min}} \left( \frac{B_0}{\delta B_{slab}} \right)^2 \left( \frac{1}{4\sqrt{\pi}} + \left( \frac{1}{\Gamma(q/2)} + \frac{1}{\sqrt{\pi(q-2)}} \right) \frac{b^{q-2}}{Q^{q-s} R^s} + \frac{2}{\sqrt{\pi(2-s)(4-s)}} \frac{1}{R^s} \right)
\]

where

\[
f_1 = \frac{2}{p-2} + \frac{2}{2-s},
\]

\[
R = \frac{P}{B_0 k_D},
\]

\[
Q = \frac{P}{B_0 k_D} \quad \text{and}
\]

\[
b = \frac{v}{2\alpha_d v_A}, \quad \alpha_d \in [0, 1]
\]
Evaluation at 1 AU (using solar minimum turbulence quantities) yields:
Results
The Perpendicular Diffusion Coefficient

\[ \kappa_\perp = \frac{\sqrt{3} va C_0 B_0 \delta B^2}{3 B_0 \sqrt{\delta B_{tot}}} \times \]

\[ \left[ \frac{\sqrt{\pi} \lambda_2 T}{p} \left( \text{Erfc}\left[\sqrt{3}X \lambda_2 T\right] - \frac{B_0 \sqrt{\frac{3}{\pi}} \lambda_2 T E_{1+p} \left[\frac{3X^2 \lambda_2^2}{2}\right]}{a \sqrt{\delta B_{tot}} \lambda_\parallel} \right) \right] \]

\[ + \sqrt{\pi} \left( \frac{a \sqrt{\delta B_{tot}} \lambda_\parallel}{B_0 \sqrt{3\pi}} \left( e^{-3X^2 \lambda_2^2} - e^{-3X^2 \lambda_2^2} \right) - \lambda_2 D \text{Erfc}\left[\sqrt{3}X \lambda_2 D\right] + \lambda_2 T \text{Erfc}\left[\sqrt{3}X \lambda_2 T\right] \right) \]

\[ + \frac{3-s}{s} \lambda_2 D (X \lambda_2 D)^{-s} \left[ \left( \frac{3s/2}{s} \sqrt{\pi} (X \lambda_2 D)^s \text{Erfc}\left[\sqrt{3}X \lambda_2 D\right] - (X \lambda_2 D)^s \text{Erfc}\left[\sqrt{3}X \lambda_2 D\right] \right) - \Gamma \left[ \frac{1+s}{2}, 3X^2 \lambda_2^2 D \right] + \Gamma \left[ \frac{1+s}{2}, 3X^2 \lambda_2^2 D \right] \right) \]

\[ + \frac{3-q/2}{q} \lambda_2 D \left( \frac{\lambda_D}{\lambda_2 D} \right)^s (X \lambda_D)^{-q} \left( \frac{3q/2}{s} \sqrt{\pi} (X \lambda_D)^q \text{Erfc}\left[\sqrt{3}X \lambda_D\right] + \Gamma \left[ \frac{1+q}{2}, 3X \lambda_2 D \right] - \Gamma \left[ \frac{1+q}{2}, 3X \lambda_2 D \right] \right) \]

where \( X = \frac{B_0}{a \sqrt{\delta B_{tot}} \lambda_\parallel} \)
Evaluation at 1 AU (using solar minimum turbulence quantities) yields:
ANALYSIS OF PERPENDICULAR MEAN FREE PATHS

Damping Model:

![Graph showing Perpendicular Mean Free Path vs Rigidity (GV)]
ANALYSIS OF PERPENDICULAR MEAN FREE PATHS

Random Sweeping Model:
CONCLUSIONS AND FUTURE WORK
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The model used for $\kappa_\parallel$ affects the values of $\kappa_\perp$. 

By approximating the power spectrum as only the energy containing range term, the calculations are greatly simplified.

In the future:

Solve $\kappa_\perp$ for dynamical turbulence, i.e.

$\gamma(k_\perp) \neq 0$. 

Investigate the effects of these results on cosmic ray transport.
The model used for $\kappa_{||}$ affects the values of $\kappa_\perp$.

The dissipation range term of $\kappa_\perp$ is negligible. Cosmic ray modulation studies that omit dissipation range effects are therefore accurate.

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The dissipation range term of $\kappa_{\perp}$ is negligible. Cosmic ray modulation studies that omit dissipation range effects are therefore accurate.

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Turbulence Quantities at 1 AU

Outer scale:
$\lambda_{2D} = 0.0074 AU$ [Weygand et al., 2011]

Turnover scale:
$\lambda_{2T} = 0.1 AU$ [Engelbrecht, 2019]

Dissipation scale:
$\lambda_D = 10^{-5} AU$ [Leamon et al., 2000]


