

# Forward Application of Propagation Path Effects in Radio Interferometry Using Measurement Equations

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**Abstract.** The Radio Interferometric Measurement Equation (RIME) is an elegant mathematical formalism that is uniquely suited for modelling both the direction-independent (DIEs) and the direction-dependent (DDEs) observational effects exhibited by existing radio interferometers (VLA, GMRT, WSRT) and upcoming instruments like SKA and its pathfinders. This paper provides a brief introduction to the RIME and proceeds to discuss how it is being implemented for predicting visibilities from the sky model in MeqTrees, a software package for radio interferometric simulation and calibration.

## 1. Introduction

A radio interferometer measures *visibilities*, which are related to the radio brightness distribution in the sky by a Fourier transform. In practice, the incoming radio signals are corrupted by various source-dependent and instrument-dependent effects along the propagation path. For an excellent introduction to radio interferometry and its techniques, refer [1]. The per-station<sup>1</sup> complex gain terms, that do not vary across the field-of-view are the direction-independent effects or DIEs. Effects like the primary beam pattern or ionospheric refraction that vary with the source direction are called direction-dependent effects or DDEs.

The conventional “open-loop” calibration procedure along with the technique of *self-calibration* [2], termed 1GC (for 1<sup>st</sup> Generation Calibration) and 2GC respectively<sup>2</sup>, can handle the DIEs effectively and help astronomers regularly reach dynamic ranges exceeding  $10^4$ . But to obtain high dynamic range images that approach the theoretical noise limit, it is necessary that the equations that characterize an observation account for the improved sensitivities and new design techniques (eg., phased arrays) of existing and upcoming radio telescopes and ultimately, deal with the DDEs. The radio interferometry measurement equation (RIME), provides a complete mathematical framework to deal with all observational effects including DDEs, unlike the approximate and less flexible pre-RIME formulations. Once a RIME is formulated, it can be solved for to predict visibilities, which can serve as simulations of an observation or as models for calibrating existing data. Such calibration methods that aim to handle DDEs are called Third Generation Calibration (3GC) techniques.

MeqTrees [3] is a software package that provides a flexible way to construct a RIME in the form of a *tree*. It can solve for any RIME, given enough constraints on the parameters to be

<sup>1</sup> A *station* is a generic term referring to one element of an interferometer array.

<sup>2</sup> Originally proposed by J. E. Noordam and O. M. Smirnov [3].

solved for. Starting with a qualitative overview of the RIME, this paper proceeds to discuss how MeqTrees implements the RIME for visibility prediction. It then discusses the UVBrick module which is used to simulate extended sources and how DDEs can be applied in the forward direction (visibility prediction from sky model) within UVBrick.

## 2. The Radio Interferometry Measurement Equation

The original 4 x 4 *Mueller* matrix formalism of the RIME was proposed by Hamaker et al [4]. Hamaker [5] rewrote the *coherency vector* that describes the visibility into a 2 x 2 *coherency/visibility matrix* so that it was more conducive for matrix multiplication with the simpler 2 x 2 *Jones* matrices [6] used to describe observational effects. This is the form of RIME that is implemented in MeqTrees and is derived in detail by Smirnov [7]. In this representation, given

$$\mathbf{v}_p = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

where,  $v_x$  and  $v_y$  are the two polarization components<sup>3</sup> measured by station  $p$ , the visibility measured by the two elements of an interferometer becomes

$$V_{pq} = \langle \mathbf{v}_p \mathbf{v}_q^\dagger \rangle = \begin{pmatrix} \langle v_{px} v_{qx}^* \rangle & \langle v_{px} v_{qy}^* \rangle \\ \langle v_{py} v_{qx}^* \rangle & \langle v_{py} v_{qy}^* \rangle \end{pmatrix} \quad (1)$$

The RIME can be written in many different forms. One useful form of writing it is to consider the sky to be composed of a set of discrete point sources. In this case, the predicted visibility or the 2 x 2 *coherency matrix*  $V_{pq}$  between two stations  $p$  and  $q$  of an interferometer is given by,

$$\begin{aligned} V_{pq} &= \mathbf{G}_p \left( \sum_{s=1}^N \mathbf{E}_{ps} X_{spq} \mathbf{E}_{qs}^\dagger \right) \mathbf{G}_q^\dagger \\ X_{spq} &= K_{sp} \mathbf{B}_s K_{sq}^\dagger \end{aligned} \quad (2)$$

$X_{spq}$  is the baseline-dependent *source coherency* which can be interpreted as the RIME for a single uncorrupted point source, with  $\mathbf{B}$  being the 2 x 2 *brightness matrix* that describes the polarized sky brightness and  $K_{sp}$ , a scalar *K-Jones* matrix that represents the geometric phase delay [7].  $\mathbf{G}_p$  and  $\mathbf{E}_{sp}$  are the Jones matrices that describe the DIEs and DDEs respectively.

For the more general all-sky case, in which the sky has a continuous brightness distribution that varies with the source direction  $\boldsymbol{\sigma}$ , the RIME becomes,

$$\begin{aligned} V_{pq} &= \mathbf{G}_p \left( \iint_{lm} \mathbf{B}_{pq} e^{-2\pi i(u_{pq}l + v_{pq}m)} dl dm \right) \mathbf{G}_q^\dagger \\ \mathbf{B}_{pq} &= \mathbf{E}_p \mathbf{B} \mathbf{E}_q \end{aligned} \quad (3)$$

The above equation provides insight into how DDEs affect the calibration process and the dynamic range of the final image. In the case of *trivial* DDEs,  $\mathbf{E}_p \equiv \mathbf{E}$ , and the *apparent* sky  $\mathbf{B}_{app} = \mathbf{E} \mathbf{B} \mathbf{E}^\dagger$  is the same for all baselines. Hence the full-sky RIME reduces to the discrete sky case, eq. (2), in which the parenthesized term is replaced with the sky coherency  $X_{pq} = \mathcal{F} \mathbf{B}_{app}$ .

In the case of *non-trivial* DDEs, the apparent sky varies with baseline and so a simple Fourier transform does not suffice. Since DDEs are multiplicative errors in the image domain,

<sup>3</sup> The two polarization components measured can be either linear or circular as long as the RIME is consistent. If necessary, 2D matrices for conversion between linear and circular co-ordinate systems can be inserted into the RIME, as discussed in [7].

by convolution theorem [8], the observed visibility can be written as the convolution of the sky coherency with the Fourier transforms of the sky-Jones matrices  $\mathbf{E}_p$  [9]:

$$\begin{aligned} X_{pq}[t] &= \mathbf{U}_p[t] * \mathbf{X} * \mathbf{U}_q^\dagger[t] \\ \mathbf{X} &= \mathcal{F}\mathbf{B}, \mathbf{U}_p[t] = \mathcal{F}\mathbf{E}_p[t] \end{aligned} \quad (4)$$

Thus the observed visibility is actually a sampling of a different  $uv$  plane for each baseline  $pq$  and time interval  $t$  and the “single common sky” assumption of traditional calibration methods falls apart. Application of non-trivial DDEs in the forward direction is discussed in section 4.

### 3. MeqTrees and RIME

The MeqTrees software system is discussed in detail in [3]. Each node in MeqTrees receives a *Request* object from its parent(s), calls on the *child* nodes if necessary, does the necessary computations and returns a *Result* object. MeqTrees uses *directed acyclic graphs* or *dags*<sup>4</sup> to represent mathematical expressions, where all nodes except the root can have one or more parent nodes. Instead of letting the nodes evaluate to a single real or complex value, MeqTrees allows *MeqNodes* to be *functions* of  $N$  variables represented over  $N$ -dimensional *grids*<sup>5</sup>. Solving for parameters  $p_1, p_2, \dots, p_k$  on which any model  $M(\mathbf{p})$  depends, is done by adapting a least-squares method to minimize the  $\chi^2$  sum and iterating until convergence is reached or a preset number of iterations are performed (see Eq. 1 in [3]):

$$\chi^2(\mathbf{p}) = \sum_{ij} w_{ij}^2 (D_{ij} - M_{ij}(\mathbf{p}))^2 \quad (5)$$

where,  $D$  represents the data and  $w_{ij}$ , the optional weights.

A formulation of RIME like eq. (2) can be easily represented in MeqTrees as shown in Fig. 1. Such a tree can be constructed for every baseline, with different E-Jones and G-Jones matrices implemented by independent subtrees. Also, as illustrated in Fig. 2, the brightness matrices (and hence the source coherency matrices) can be represented as a mixture of point sources, Gaussian components, FITS images (refer Sec. [3.2]) and shapelets, with arbitrary subtrees evaluating to each source component  $X_s$ . This modularity is one of the biggest strengths of MeqTrees.

#### 3.1. Simulation and Calibration within MeqTrees

Given an initial sky model and the instrument parameters, the RIME predicts the visibilities pertaining to an observation. Depending on whether simulation or calibration needs to be performed, the form that the RIME takes could vary. For simulation, *explicit* forms of RIME are used wherein the known physical effects are represented by separate Jones matrices (for a description of various Jones matrices, refer [3]). Different methods (subtrees) can be employed to compute the different Jones terms involved.

During calibration, the predicted visibilities are used as models  $M(\mathbf{p})$ , that are fitted to the data  $D$ , as in eq. (5). In such cases, the Jones terms solved for could end up subsuming multiple corrupting effects and it is not necessary that they be separable, since we are only interested in correcting for the overall phase error. These are called *phenomenological RIMEs* [9]. The differential gains method that is employed by Smirnov for calibration of the 3C147 field in [10] is an example of a phenomenological RIME.

<sup>4</sup> A directed graph with no directed cycles. Strictly speaking, MeqTrees is *MeqDags*.

<sup>5</sup> In radio astronomy, the independent variables are usually time  $t$  and frequency  $\nu$ .

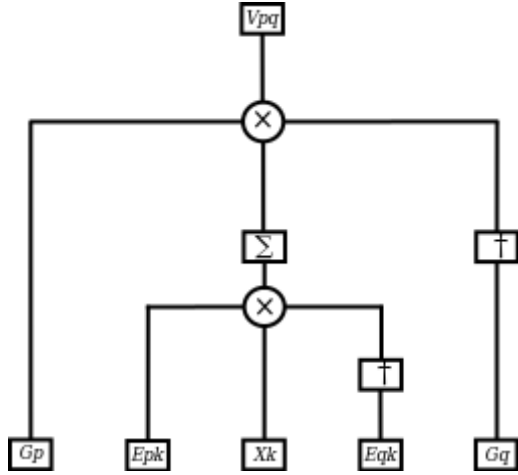


Figure 1: Implementing the RIME eq. (2) in MeqTrees. The summation runs over the index  $k$  and  $pq$  represents one baseline.

### 3.2. The UVBrick

The *UVBrick* is a collection of MeqTree nodes that are used to predict the visibilities of extended sources or of multiple point sources. MeqTrees performs a Discrete Fourier Transform to predict  $uv$  domain data for bright, point sources. For fainter extended sources, the preferred approach is to use patch images on which a more computationally efficient Fast Fourier Transform [11] can be performed, so that all sources in the field of view are treated simultaneously. To avoid aliasing effects the input image is zero-padded.

*MeqFFTBrick.* The Local Sky Model consists of a set of patch images, with one image per frequency/brightness plane. The MeqFFTBrick node performs a two-dimensional FFT from the  $lm$  plane to the  $uv$  domain for each slice through the frequency and brightness axes. The FFT is performed using the FFTW library [12].

*MeqUVInterpolWave.* The MeqUVInterpolWave node has two children; one that receives all the  $uv$ bricks created by MeqFFTBrick and one that reads in the measured  $uv$  tracks from the observation (a Measurement Set). This node then interpolates for the sampled  $uv$  points from the  $uv$  grid (Fig. 3) for each requested frequency, by linearly interpolating between the appropriate  $uv$ bricks. This “degridding” is performed using spherical convolutional functions [13]. The interpolation in frequency enables the UVBrick to take any real spectral variation present in the local sky model into consideration.

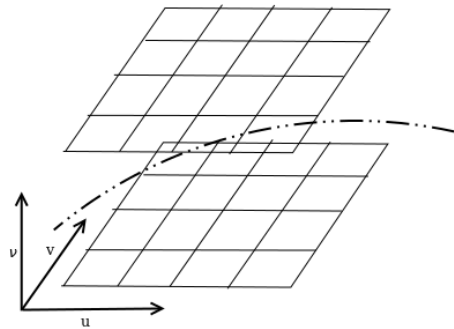


Figure 3: Degridding and frequency interpolation in the UVBrick. The dotted line represents the sampled  $uv$  points at which the *bricks* (grids) are convolved with a spherical function. The vertical axis is the frequency  $\nu$ .

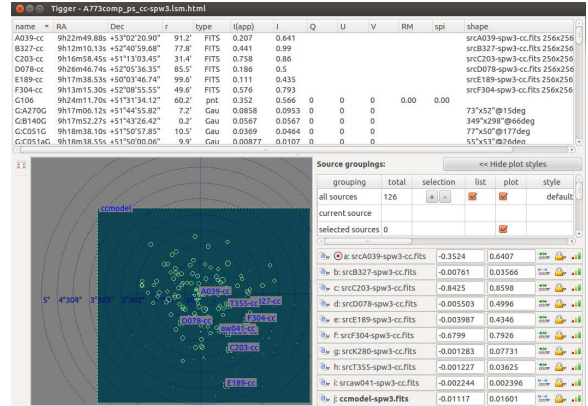


Figure 2: A sample local sky model consisting of Gaussians, FITS images and point sources visualized with Tigger.

#### 4. Applying DDEs within UVBrick

Different approaches have been proposed for correcting for non-trivial (both known and unknown) DDEs, as discussed in [9]. As mentioned earlier, it is possible in MeqTrees to use the more accurate DFT-based approach in conjunction with the computationally more efficient FFT-based approach to achieve a fair trade-off between computational cost and accuracy. The methods discussed here are those which use convolutional functions for handling DDEs.

##### 4.1. W-Projection

One approach that makes use of convolution to handle the non-coplanar baselines problem is the *W-Projection* algorithm proposed by Cornwell et al. [14]. The visibility  $V$  and the sky brightness  $I$  are related by (see [1]),

$$V(u, v, w) = \int \frac{I(l, m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} dl dm \quad (6)$$

where,  $u, v, w$  are the baseline components measured in units of wavelength  $\lambda$  of the incoming radiation and  $l, m$  are the direction cosines of the source direction with respect to the phase centre. Eq. (6) is just another way of writing eq. (3) without explicitly writing out the Jones matrices. When the value of  $2\pi w(\sqrt{1-l^2-m^2}-1)$  is not  $\ll 1$ , a simple 2D F.T is not enough. A single interferometer in the “ $uvw$  space” no longer measures a single Fourier component in the “Fourier space”. Frater and Docherty [15] considered the case of a 2D interferometer located at  $w \neq 0$  and derived the relation for projecting it onto the  $w = 0$  plane using a convolutional function approach. Cornwell et al. [14] extend this to show that reprojection from/to any  $w$ -plane to/from  $w = 0$  plane is possible:

$$\begin{aligned} V(u, v, w) &= G_w * V_{w=0} \\ G_w &= \mathcal{F}(e^{-2\pi i w(\sqrt{1-l^2-m^2}-1)}) \end{aligned} \quad (7)$$

Fig. 4 compares the standard Fourier transform with an image of the same field after *W-Projection*.

Since a convolution already exists within UVBrick for degriding, it can be extended to include the convolution function  $G_w$  to account for the  $W_p$  term in  $uvw$  space. The *MeqNBrick* node that is in development aims to handle these  $w$ -term effects, by pre-computing a set number of convolution kernels to be used by *MeqUVInterpolWave* while degriding.

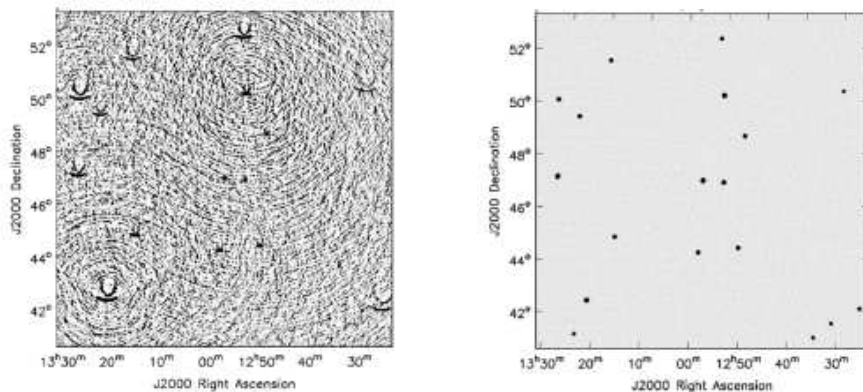


Figure 4: Results of a 74 MHz simulation [14]. (a) shows the result of a standard Fourier transform and (b) shows the same image with *W-projection* applied (128  $G_w$  planes).

#### 4.2. AW-Projection

The W-Projection algorithm was generalized to *AW-Projection* by Bhatnagar et al [16], in which they show that similar to how the  $W_p$  term is handled using  $G_w$  [eq. (7)], the convolution functions  $U_p$  [eq. (4)] can be applied to the source coherency  $X$  during degriding (in the forward direction) and corrected for during imaging<sup>6</sup>. Within UVBrick, the spheroidal function used for degriding can incorporate the necessary  $U_p$  matrices as it can the  $G_w$  functions. But unlike W-Projection, where the  $G_w$ 's can be computed once before degriding and stored, the convolution functions  $U_p$  are not simple analytical expressions and need to be computed on-the-fly by child nodes and supplied to the node that does the degriding. The UVBrick is expected to be able to apply arbitrary DDEs by implementing the AW-Projection algorithm in the future.

### 5. Conclusion

This paper gave a brief introduction to measurement equations and the MeqTrees software system and how the RIME is implemented within MeqTrees. It also discussed how the MeqTrees UVBrick module predicts visibilities from a local sky model and can incorporate corrections for DDEs. As the understanding on applying corrections for non-trivial DDEs improves and new techniques and algorithms are proposed, they can be implemented within MeqTrees fairly easily and tested. More information on MeqTrees and measurement equations can be obtained from [17], [18], [19] and the MeqTrees website: <http://www.astron.nl/meqwiki/MeqTrees>.

The author would welcome discussions from anyone interested in 3GC in radio astronomy.

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<sup>6</sup> In the reverse direction, this can be done for DDEs that have Jones matrices that are approximately unitary and have limited support in the Fourier domain [9], [16].