Unsharp mesaurement in Quantum Mechanics and its application to monitor Rabi Oscillations

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Abstract. In recent years a generalized notion of quantum mechanical observables has been developed in terms of positive operator valued measure (POVM). The POVM formalism of observables captures features of quantum mechanics in a more comprehensive way than the standard formalism. Certain POVMs can be interpreted as unsharp quantum measurements. This article reviews the notion of unsharp measurement, its utility and experimental realization.

1. Introduction

Usual quantum measurements are projective measurements which project the initial state of a system onto one of the eigenstates of the observables being measured. For example in a measurement for spin along direction $\hat{r}$, the projectors onto the eigenstates are:

$$\hat{P}_{\pm} = \frac{1}{2}[\mathbb{I} \pm \hat{r}.\vec{\sigma}], \tag{1}$$

$\mathbb{I}$ denotes the identity operator and $\vec{\sigma} = (\hat{\sigma_x}, \hat{\sigma_y}, \hat{\sigma_z})$ is the usual Pauli-operator.

However, further progress had shown that the most general quantum measurements are given by positive operator valued measures (POVM) [1]. These generalized measurements allow us to describe any measurement that can be performed within the limits of quantum theory.

The POVM formalism of observables captures features of quantum mechanics in a more comprehensive way than the standard formalism. It is worth mentioning here that Bell could construct a Hidden Variable Theory for two dimensional quantum system by using standard observables but it has been shown recently that if one uses the formalism of generalized observables (i.e. the POVM formalism), then even for two dimensional quantum system, Gleasons theorem as well as the Kochen-Specker theorem hold [2, 3]. This formalism creates the possibility of certain joint measurements of complementary observables like position and momentum; spin along two different directions, etc., it has also been used to show that the CHSH expression should be bounded by $2\sqrt{2}$ for quantum systems to avoid superluminal signalling in quantum mechanics [4]. On the other hand, it is turning out to be very useful in controlling quantum systems [5, 6].

2. Generalized measurement in quantum mechanics

In this more general framework of quantum theory, the states of a quantum system are represented by positive trace class operators. Any observable is represented by a collection of positive operators $\{E_i\}$ where $0 \leq E_i \leq \mathbb{I}$ for all i and $\sum E_i = \mathbb{I}$, $\mathbb{I}$ being the unit operator on the Hilbert space associated with the system. In a measurement of this observable for the state ρ (say), the probability of occurance of the ith result is given by $Tr[\rho E_i]$. Unlike projective measurement, knowing the operators $\{E_i\}$ is, in general, not enough to determine the state of the system after measurement; we further need to know the operators $\hat{M}_i$'s constituting the POVM elements $\{E_i\}$. As an example, let $E_i = \sum_j M_{ij}^\dagger M_{ij}$, then after the outcome i, the state is $\rho \mapsto \rho' = \frac{1}{Tr[\rho E_i]} \sum_j M_{ij} \rho M_{ij}^\dagger$. This measurement, does not preserve the purity of states, in general, unless there is a single M_{ij} for each E_i and in that case the post measured state is given by $|\psi'\rangle = \frac{M_i|\psi\rangle}{\sqrt{\langle\psi|M_i^\dagger M_i|\psi\rangle}}$.

Let us consider the following opeartors:

$$\begin{aligned} \hat{M}_0 &= \sqrt{p_0}\, \hat{P}_+ + \sqrt{1-p_0}\, \hat{P}_- \\ \hat{M}_1 &= \sqrt{1-p_0}\, \hat{P}_+ + \sqrt{p_0}\, \hat{P}_- \end{aligned} \tag{2}$$

related via

$$M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{I}, \tag{3}$$

and $0 \leq p_0 \leq 0.5$. The positive opeartors $M_i^\dagger M_i$, $(i \in 0, 1)$ constitute POVM elements and are interpretaed as unsharp dichotomic observables (for example, spin observable of a spin-1/2 particle, energy of a two-level system etc.) [7, 8, 9], the measurement strength is parametrized by the quantity $\Delta p = (1 - p_0) - p_0 = 1 - 2p_0$. For $\Delta p = 1$, it represents the usual projective (sharp) measurement. The M_0 and M_1 (given in (2) are diagonal in the basis corresponding to the projectors (1). This assures that post-measurement state is again a superposition of this basis element if we start with such a superposition and thus the post measurement state is not qualitatively different from the pre-measurement state. Moreover, these Kraus operators are minimal, in the sense that they cannot be further decomposed as a product of a unitary operator (other than the identity) and another operator. This minimality assures no unnecessary disturbance of the state due to the measurement as an unitary evolution does not add to our knowledge about the system.

3. Realization and application of unsharp measurement in monitoring Rabi Oscillations

Any POVM can be realised, in principle, by allowing the measured system to interact with an ancilla, and then doing a projective measurement on the ancilla. As will be shown elsewhere two ions of different species interacting with laser light and moving inside a linear trap provide a realistic system to implement the generalized measurement given by (2) on a qubit.

In order to monitor the dynamics of a qubit undergoing Rabi oscillations, we generally need an ensemble prepared in the initial state of the qubit. The oscillating Rabi-probabilities at a particular time t are then determined by performing projective measurements (at that time) of a dichotomic observable, having the two states of the qubit as its eigenstates, on each member of the ensemble. For knowing these probabilities at a different time, the procedure is repeated from the begining. If, instead, we are supplied with only a single qubit, the above procedure of visualization fails. In such a situation, evolution of the system is tracked by performing a sequence of above said measurement (2) on the qubit with period τ [5]. It is assumed that the time it takes to execute each measurement is negligible compared to all other timescales of the system. In between measurements the system evolves through the Hamiltonian $H_R = \hbar \frac{\Omega_R}{2} \sigma_x$ (written in a frame rotating at the transition frequency (Ω_R) between its two internal states). At

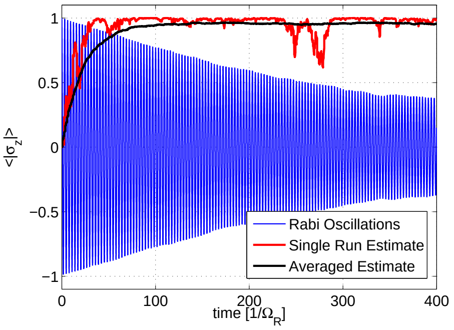

Figure 1. Wavefunction estimation in the presence of dephasing noise. The red line shows the estimation fidelty for a single run, while the black line gives the expected fidelity obtained by averaging over 1000 runs. The blue line demonstrates that the Rabi oscillations monotonically lose coherence in the absence of unsharp measurements [5] .

$t = N\tau$, i.e., after N measurements the system is, up to the appropriate normalization constant, in the state

$$|\psi(N\tau)\rangle = M_{n_N} U M_{n_{N-1}} ... M_{n_1} U |\psi\rangle., \qquad (4)$$

where $U = \exp\{-\frac{i}{\hbar} H_R \tau\}$. The overall strength of a sequence of measurements depends on the strength of a single measurement ($\Delta p = 2p_0 - 1$), together with its frequency. The influence of a sequential unsharp measurement on the state of the qubit is parametrized by a quantity $\gamma = \frac{(\Delta p)^2}{\tau}$ which can be identified by the inverse of the decoherence time [10] of the system [11] when averaging over the measurement results. The sequential measurement has a stonger influence on the state for a larger γ. The Hamiltonian dynamics of the system is characterized by the Rabi-frequency. For γ much larger than the Rabi-frequency, the disturbance by the measurement causes quantum jumps (a zeno type effect). Thus, in order not to disturb the original dynamics too much, we need to keep the measurement strength (γ) well below the Rabi-frequency.

In order to estimate the state of the system the same sequence of operators corresponding to the measured outcomes in Eq. (4) are applied to an initial guess $|\psi'\rangle$ which can be taken as an arbitrary state vector on the Bloch sphere and in between measurements the estimated state is assumed to evolve through the Hamiltonian H_R [12]. Interestingly, the estimate closely approximates the true state within a few Rabi cycles [5]. The idea of estimating the state by updating the initial guess with the outcomes obtained by doing continuous measurement on the actual system was first introduced by Diosi et al [12]. It was also shown there that ultimately the estimate converges to the actual state.

We gave an scheme to monitor the dynamics of a qubit without taking into account the noises present. Interestingly, the above estimation strategy still works with a reasonably high fidelity even in presence of noise. To estimate the state, the abovesaid sequence of unsharp measurement is again performed on the system and the same sequence of operators corresponding to the measured outcomes in Eq. (4) are applied to an arbitrary initial guess, but in between measurements the estimated state is assumed to evolve only through the Hamiltonian H_R since the experimenter does not know what the instantaneous values of the noise fields are. Interestingly enough, the estimate quickly approaches the real state with a reasonably high fidelity (fig 1)(though this time the fidelity is not unity) [5].

4. Conclusion

In conclusion, we have reviewed here the formulation of unsharp measurement in quantum mechanics and its utility in monitoring the dynamics of a single qubit undergoing Rabi oscillations.

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