Oscillating cosmological correlations in $f(R)$ gravity

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Abstract. The purpose of this paper is to investigate the oscillatory behavior of the universe through a Schrödinger-like Friedmann equation and a modified gravitational background described by the theory of $f(R)$ gravity. The motivation for this stems from the observed periodic behaviour of large-scale cosmological structures when described within the scope of the general theory of relativity. The analysis of the modified Friedmann equation for the dust epoch in power-law $f(R)$ models results in different behaviors for the wave-function of the universe. These oscillatory instances of the wave-function point towards a possible ordered pattern in the manner in which clustering occurs during the universe's evolution.

1. Introduction

According to the Cosmological Principle (CP), matter in the universe is homogeneously and isotropically distributed; a principle that is confirmed by observations on sufficiently large scales ($1000h^{-1}$Mpc). However, observations show that the CP breaks on scales of the order $\sim 100h^{-1}$ Mpc and below, and the clustering property of cosmological objects (galaxies, clusters, superclusters, filaments) shows there exists some sort of hierarchy. Thus, there are suggestions that the distribution of galaxies is not random and that some fundamental mechanism has led to the formation of large-scale structure. Such an idea is built from the notion that the density inhomogeneities that are the primordial causes of the formation of structure in the universe are as a result of quantum fluctuations generated during inflation. One proposal is cosmological solutions with an overall Friedmannian expanding behaviour, corrected by small oscillatory regimes [1]. Given the cosmological scale factor $a(t)$ and redshift z, we have the Hubble parameter H given by

$$\frac{\dot{a}}{a} = \frac{\Theta}{3} \equiv H = -\frac{\dot{z}}{1+z} , \tag{1}$$

wherein Θ represents the cosmological (volume) expansion parameter. Oscillations at a particular redshift can be considered as some sort of quantization [1, 2] and all quantities containing H or z have to oscillate. These oscillations affect several observational quantities, such as the number count of galaxies

$$\frac{dN}{d\Omega dL dz} = n(L, t_0) a_0^2 H^{-1} d^2 , \tag{2}$$

where dN is the number of galaxies in the solid angle $d\Omega$ having redshift between z and $z + dz$ and luminosity between L and $L + dL$,$n(L, t_0)$ that represents the number density of galaxies with luminosity L that an observer sees at time t_0, a_0 is the value of the cosmological scale

factor today and d is the comoving distance defined as $d = \int_t^{t_0} \frac{dt}{a}$.
There have also been recent attempts to link gravitation with quantization, largely motivated by the need to unify two of theoretical physics' most fundamental theories into one overarching framework. There are generally two main approaches in this endeavor:

- The whole universe as a quantum system of co-existing and non-interacting universes [4, 5]
- The universe as a classical background: where primordial quantum processes gave rise to the current macroscopic structures [3, 6]

Following Capozziello [1] and Rosen [7], one can recast the cosmological Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{9}\Theta^2 = \frac{\mu}{3} - \frac{k}{a^2} \tag{3}$$

as some sort of a Schrödinger equation (SE). To do so, we can rewrite the above equation as the equation of motion of a "particle" of mass m:

$$\frac{1}{2}m\dot{a}^2 - \frac{m}{6}\mu a^2 = -\frac{1}{2}mk\ , \tag{4}$$

where $\Theta \equiv 3H$, μ and k are, respectively, the cosmological (volume) expansion parameter, the energy density and spatial curvature of the universe. The total energy E of the particle can be thought of as being the sum of the kinetic T and potential V energies:

$$E = T + V \tag{5}$$

where

$$T = \frac{1}{2}m\dot{a}^2\ , \quad V = -\frac{1}{6}m\mu a^2\ , \quad E = -\frac{1}{2}mk\ . \tag{6}$$

One can also rewrite the Raychaudhuri (acceleration) equation

$$\frac{\ddot{a}}{a} = -\frac{1}{2}(\mu + 3p) \tag{7}$$

in a way that mimics the equation of motion of the particle, otherwise given by

$$m\ddot{a} = -\frac{dV}{da}\ , \tag{8}$$

where p is the isotropic pressure, related to the energy density of a perfect fluid through the equation of state parameter w as $p = w\mu$. The particle's momentum and Hamiltonian are defined, respectively, as

$$\Pi \equiv m\dot{a}\ , \qquad H \equiv \frac{\Pi^2}{2} + V(a)\ . \tag{9}$$

From the "first quantization" scheme, we have

$$\Pi \rightarrow -i\hbar\frac{\partial}{\partial a}\ . \tag{10}$$

Thus the SE for the wavefunction $\Psi = \Psi(a, t)$ is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial a^2} + V(a)\Psi\ . \tag{11}$$

We can think of m as the mass of a galaxy, and $|\Psi|^2$ as the probability of finding the galaxy at $a(t)$ or at a given redshift

$$1 + z = \frac{a_0}{a} \,, \tag{12}$$

and thus, in the language of quantum physics, $\Psi = \Psi(z,t)$ defines the probability amplitude to find a given object of mass m at a given redshift z, at time t. The stationary states of energy E are given by

$$\Psi(a,t) = \psi(a) e^{-iEt/\hbar} \,, \tag{13}$$

and the time-independent Schrödinger equation (TISE) reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{da^2} + V\psi = E\psi \,. \tag{14}$$

2. $f(R)$ Gravitation

$f(R)$ models are a sub-class of *fourth-order* theories of gravitation, with an action given by [9] [1]

$$\mathcal{A}_{f(R)} = \tfrac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m] \;, \tag{15}$$

where R, g and $\mathcal{L}_m$ are the Ricci scalar, the determinant of the metric tensor, and the matter Lagrangian. The $f(R)$-generalized Einstein field equations can be given by

$$f' G_{ab} = T^m_{ab} + \tfrac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f' \,. \tag{16}$$

Here primes symbolize derivatives with respect to R, whereas G_{ab} and T^m_{ab} are the Einstein tensor and the energy-momentum tensor of matter respectively. These models provide the simplest generalizations to GR, and come with an extra degree of freedom. The cosmological viability of the models can be determined through observational and theoretical constraints. Some generic viability conditions on f include [8]:

- To ensure gravity remains attractive

$$f' > 0 \;\; \forall R \tag{17}$$

- For stable matter-dominated and high-curvature cosmological regimes (nontachyonic scalaron)

$$f'' > 0 \;\; \forall R \gg f'' \tag{18}$$

- GR-like law of gravitation in the early universe (BBN, CMB constraints)

$$\lim_{R\to\infty}\frac{f(R)}{R} = 1 \Rightarrow f' < 1 \tag{19}$$

- At recent epochs

$$|f' - 1| \ll 1 \tag{20}$$

The matter-energy content of a universe filled with a perfect fluid is specified by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \,. \tag{21}$$

The background curvature and total perfect fluid thermodynamics is described by [10]

$$\begin{aligned}
\mu_R &= \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R}\right], \\
p_R &= \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\Theta f''\dot{R}\right] \,, \\
\mu &\equiv \frac{\mu_m}{f'} + \mu_R \,, \quad p \equiv \frac{p_m}{f'} + p_R \,.
\end{aligned} \tag{22}$$

[1] In geometrized units: $c = 8\pi G \equiv 1$.

3. The Cosmological Schrödinger Equation

In $f(R)$ gravity, the Raychaudhuri equation generalizes to

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\Big(\frac{\mu_m}{f'} + \mu_R + \frac{3p_m}{f'} + 3p_R\Big) \tag{23}$$

which in terms of the expressions for μ_R, p_R and the trace equation ($R = \mu - 3p$) can be simplified as

$$\frac{\ddot{a}}{a} = -\frac{1}{6f'}\Big(2\mu_m - f - 2\Theta f''\dot{R}\Big) \ . \tag{24}$$

Similarly, the corresponding modified Friedmann equation in $f(R)$ gravity is given by

$$\frac{1}{9}\Theta^2 + \frac{k}{a^2} = \frac{1}{6f'}\Big(2\mu_m + Rf' - f - 2\Theta f''\dot{R}\Big) \ . \tag{25}$$

But for FLRW models, it is also true that

$$\frac{1}{9}\Theta^2 + \frac{k}{a^2} = \frac{1}{6}R - \frac{\ddot{a}}{a} \ , \tag{26}$$

one can therefore re-write equation (4) as

$$\frac{1}{2}m\dot{a}^2 - \frac{1}{2}\Big(\frac{R}{6} - \frac{\ddot{a}}{a}\Big)ma^2 = -\frac{1}{2}mk \ , \tag{27}$$

with the potential

$$V(a) = -\frac{1}{2}\Big(\frac{R}{6} - \frac{\ddot{a}}{a}\Big)ma^2 \ . \tag{28}$$

Now rearranging the TISE (14) for $f(R)$ gravity yields

$$\frac{d^2\psi}{da^2} = -\frac{2m}{\hbar^2}\Big[E - V(a)\Big]\psi = \Big[C - \frac{m^2}{6f'\hbar^2}\Big(2\mu_m + Rf' - f - 2\Theta f''\dot{R}\Big)a^2\Big]\psi \ , \tag{29}$$

where $C \equiv \frac{m^2k}{\hbar^2}$.

4. Oscillating Solutions

Let us now consider power-law $f(R)$ models of the form

$$f(R) = R^n \ , \tag{30}$$

admitting scale factor solutions

$$a(t) = a_0 t^{\frac{2n}{3(1+w)}} \ . \tag{31}$$

For dust ($w = 0$) models, we get

$$\Theta = \frac{2n}{t} \ , \quad R = \frac{4n(4n-3)}{3t^2} \ , \quad \mu_m = \frac{\mu_0}{a^3} \ . \tag{32}$$

Here a_0 and μ_0 are integration constants that can be normalized to unity when considering current values of the scale factor and the energy density of matter. Thus, for such models, the TISE (29) takes the form

$$\frac{d^2\psi}{da^2} = \Bigg\{C - \frac{m^2}{3n\hbar^2a^3}\left[\frac{4n(4n-3)}{3a^{\frac{3}{n}}}\right]^{1-n} + \frac{4nm^2(4n-3)}{18\hbar^2a^{\frac{3-2n}{n}}} - \frac{4m^2(4n-3)}{18\hbar^2a^{\frac{3-2n}{n}}} + \frac{32nm^2(n-1)}{24a^{\frac{3-2n}{n}}}\Bigg\}\psi \ . \tag{33}$$

For $n = 1$, the above equation reduces to

$$\frac{d^2\psi}{da^2} = \left[C - \frac{B}{a}\right]\psi \ , \tag{34}$$

where $B \equiv \frac{m^2}{3\hbar^2}$ and we recover the GR solutions [1] obtained by Capozziello et al. For example, for a flat universe, $C = 0$ and we get a combination of Bessel functions as the general solution:

$$\psi\,(a) = C1\,\sqrt{a}J_1\left(2\sqrt{-B}\sqrt{a}\right) + C2\,\sqrt{a}Y_1\left(2\sqrt{-B}\sqrt{a}\right) . \tag{35}$$

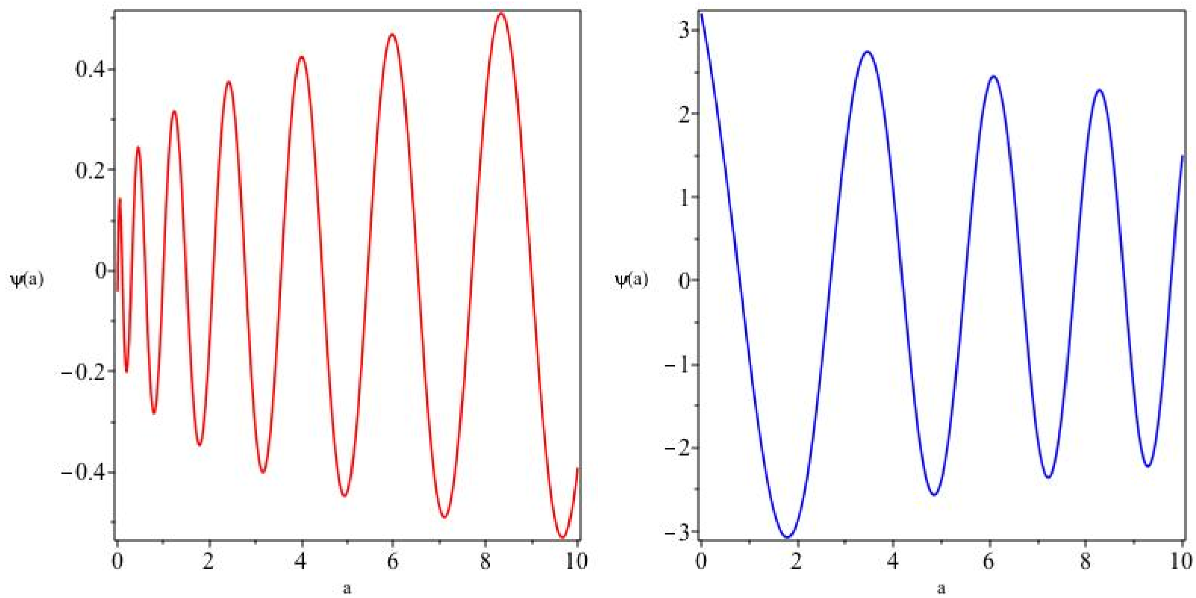
For $n = 3$, it can be shown that

$$\frac{d^2\psi}{da^2} = \left[C - Ba\right]\psi \ , \tag{36}$$

and the corresponding solutions are Airy functions of the form

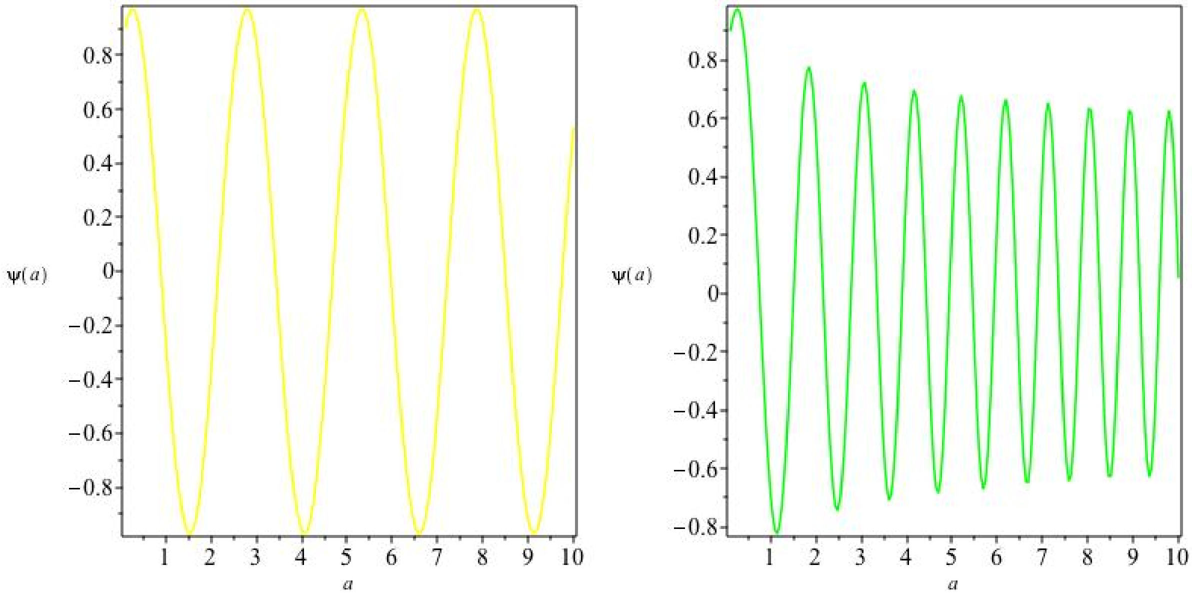
$$\psi\,(a) = C3\ \mathrm{Ai}\left(\frac{C - Ba}{B^{2/3}}\right) + C4\ \mathrm{Bi}\left(\frac{C - Ba}{B^{2/3}}\right) . \tag{37}$$

Figure (1) below shows the oscillatory behaviour of such exact-solution wavefunctions. Our investigations for the power-law models suggest that exact solutions are possible only for $n = 1$ and $n = 3$. Also, our numerical computations show no oscillatory behaviour of the solutions for $n < 1$, whereas oscillating solutions for $n = 1.5$ and $n = 2$ are presented in figure (2).



(a) Oscillating wavefunction for $n = 1$.　**(b)** Oscillating wavefunction for $n = 3$.

Figure 1: Exact solutions for $n = 1$ and $n = 3$.



(a) Oscillating wavefunction for $n = 1.5$. **(b)** Oscillating wavefunction for $n = 2$.

Figure 2: Numerical solutions for $n = 1.5$ and $n = 2$.

5. Conclusion

A breaking of homogeneity and isotropy on small scales with oscillating correlations between galaxies can be achieved with a Schrödinger-like equation. This work reproduces existing GR solutions and provides an even richer set of solutions for $f(R)$ gravity models, thus providing possible constraints on such models using observational data. For the power-law $f(R)$ model considered in this work, exact solutions have been obtained for $n = 1$ and $n = 3$ in the flat FLRW background, as well as numerical solutions for the $n = 1.5$ and $n = 2$ dust scenarios. A more detailed analysis of such oscillatory solutions with more viable $f(R)$ models and under more realistic initial conditions is currently underway.

Acknowledgments

NN and HS acknowledge the Center for Space Research of North-West University for financial support to attend the 63rd Annual Conference of the South African Institute of Physics. NN acknowledges funding from the National Institute of Theoretical Physics (NITheP). HS and AA acknowledge that this work is based on the research supported in part by the National Research Foundation (NRF) of South Africa.

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