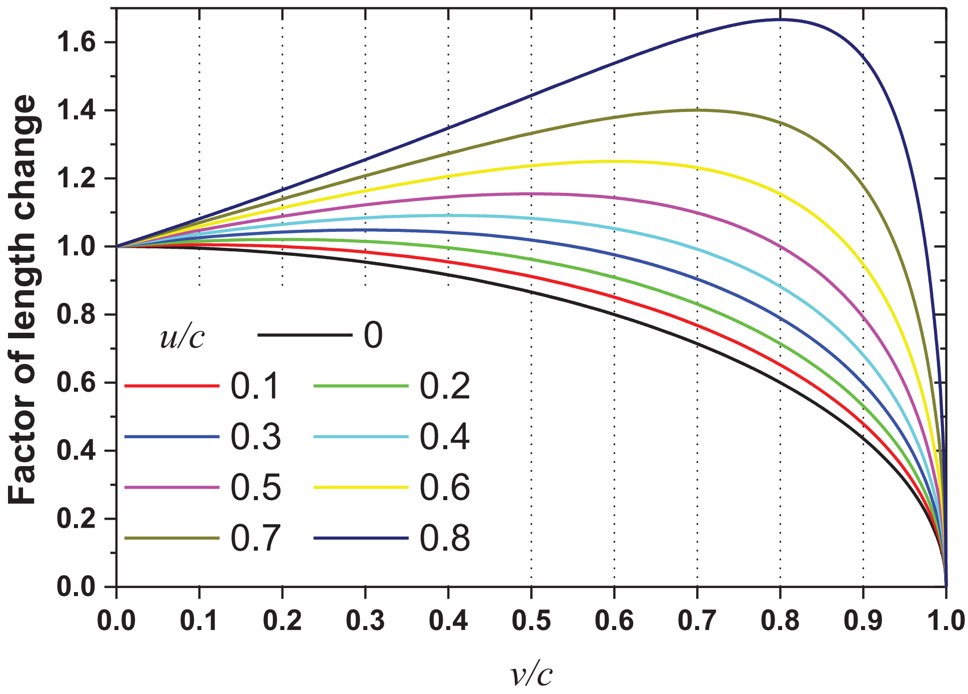

Figure 1. The factor of the length change as a function of the speed (v/c) of the new reference frame $\bar{S}$, for various speeds u/c of the object in the initial frame S.

For larger speeds a length contraction relative to the proper length L_0 occurs and there is some speed $v > u$ for which $\bar{L} = L$ again, i.e. when there is no change in the observed length despite the change in the observer's inertial reference frame. Some analysis reveals that this occurs when

$$v = \frac{2c^2}{c^2 + u^2} u \qquad \ldots (8)$$

as may readily be verified by substitution into equation (6). It may be noted that an alternative manner to obtain this result is to simply set $\bar{u} = -u$ in equation 3.

3. Transformation of the electric field considered as a consequence of the length transformation

This general length transformation can, for example, be used to simply derive the transformation of the transverse component of an electric field for a change in reference frame. Consider a parallel plate capacitor with plates of equal but opposite surface densities $\pm\sigma$ lying parallel to the XZ plane and moving with speed u in the X direction. Ignoring edge-effects, this produces an electric field $E_y = \frac{\sigma}{\varepsilon_0}$ and a magnetic field $B_z = \mu_0 \sigma u$ in the region between the plates [2].

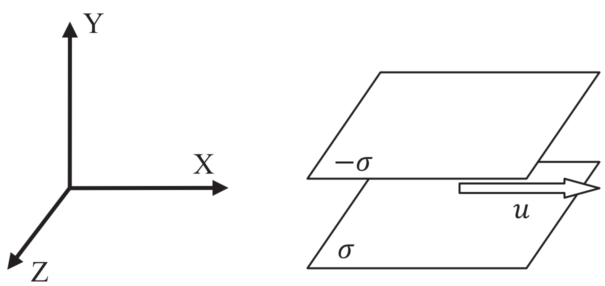

Figure 2. Moving parallel plate capacitor in original reference frame.

If the observer now moves along the X direction at speed v, there is a length transformation in the X direction i.e. contraction by the factor $\gamma_v(1 - uv/c^2)$, while the transverse Y and Z lengths are not affected. This does not affect the general geometry of the system and in the new frame

$$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0} = \frac{\sigma\gamma_v(1 - uv/c^2)}{\varepsilon_0} \qquad \dots (9)$$

where the surface charge density has increased by the factor of the length contraction in the X direction. Therefore $\bar{E}_y = \gamma_v\left[\frac{\sigma}{\varepsilon_0} - \frac{\sigma uv}{\varepsilon_0 c^2}\right]$. The first term in the bracket is the electric field in the original reference frame. Since $\frac{1}{\varepsilon_0 c^2} = \mu_0$ the second term is $\mu_0\sigma uv$ and so

$$\bar{E}_y = \gamma_v\left[E_y - vB_z\right]. \qquad \dots (10)$$

The second term in the bracket, involving the magnetic field, is therefore a direct consequence of the additional factor in equation 6 besides the Lorentz contraction for the length transformation. This derivation is given, for example, by Griffiths in his textbook *Introduction to Electrodynamics* [2]. However, the length transformation is not made explicit and the derivation is therefore significantly more complicated than necessary, which detracts from the presentation and obscures the focus on the physical concepts. The length transformation is expected to be a useful concept in many problems involving special relativity.

4. Conclusion

A description of the relativistic length transformation has been provided which is more general than the well known Lorentz contraction, which may be considered as a particular case. From the derived result a change in the observer's inertial reference frame may also result in a length increase or there may be no change in the length despite the change of the observer's inertial reference frame. Use of the relativistic length transformation, where appropriate, can simplify the physics both conceptually and mathematically, which has been illustrated here for a derivation of the transformation of the transverse component of an electric field for a change in reference frame.

References

[1] Beiser A 2003 *Concepts of Modern Physics* 6th Ed (New York: McGraw-Hill)
[2] Griffiths D J 2017 *Introduction to Electrodynamics* 4th Ed (Cambridge: Cambridge University Press)