Full phase space simulation of the relativistic Boltzmann equation in the context of heavy-ion collisions

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Abstract. Relativistic hydrodynamics has been the tool of choice to simulate the dynamics of the quark-gluon plasma produced in heavy-ion collisions. Despite the success of hydrodynamics, it has several shortcomings stemming from the fact that it assumes a system close to equilibrium. An alternative to hydrodynamics is solving the Boltzmann equation, which describes the evolution of the full distribution function of the system without the close to equilibrium requirement. The Boltzmann equation, however, has hitherto proved computationally intractable. By using a novel algorithm, and leveraging the computational power of graphics processing units, we numerically integrate the Boltzmann equation in the relaxation time approximation.

1. Introduction

Recent heavy-ion experiments have confirmed the existence of a deconfined plasma of quarks and gluons, the Quark Gluon Plasma (QGP) [1, 2]. This provides an excellent experimental opportunity to study the dynamics of the strong nuclear force, which due to confinement, has historically proved challenging.

The phenomenology of the QGP has most extensively been studied using the methods of viscous hydrodynamics [3]. Hydrodynamics, however, has limitations, most notably, it only tracks the evolution of the energy-momentum tensor, and not the more microscopic, distribution function.

Thus hydrodynamics would be unable to predict certain phenomena, for example, the possible formation of a Bose-Einstein condensate, as has been suggested by [4]. Given these considerations, we would like to go beyond hydrodynamics and have access to the full distribution function. In order to study the evolution of the distribution function, we would have to solve the Boltzmann equation, given by

$$\frac{d}{dt}f(x,p,t) = \mathcal{C}[f(p,t)] - v\nabla f(x,p,t), \tag{1}$$

where $f(x,p,t)$ is the distribution function evaluated at the point x, momentum p at time t. The $\boldsymbol{v}\nabla f(x,p,t)$ term describes the flow of matter through space. $\mathcal{C}[f(p),t]$ is the so-called collision term which is a functional that describes the interaction between different particles, and in principle requires us to determine interaction probabilities using Quantum Chromodynamics (QCD), as our goal is to simulate gluon interactions.

There have been attempts to solve the Boltzmann equation, both analytically and numerically [5, 6, 7]. These attempts have, however, assumed highly symmetric initial conditions, typically assuming exact boost-invariance and cylindrical symmetry. We, however, know that boost-invariance is just an approximate symmetry, and we would want to consider deviations from exact boost-invariance. We also know that they are not cylindrically symmetric [8].

Our goal will be to solve the Boltzmann equation with these symmetry requirements relaxed.

We will, however, make a simplifying assumption regarding the collision term, instead of using a highly non-trivial collision term derived from QCD, we will rely on a simpler phenomenological model called the relaxation time approximation.

We will furthermore exclusively consider a plasma of gluons at this stage, neglecting quarks as they cannot participate in Bose-Einstein condensation.

2. The Relaxation Time approximation

The relaxation time approximation (RTA) is a staple of statistical physics and is conceptually very simple. We know by the Boltzmann H-theorem that the Boltzmann equation will, regardless of the details of the collision term, asymptotically drive the system into its equilibrium distribution. The RTA simply assumes that the system will asymptotically approach equilibrium with some characteristic relaxation time τ_{rel}.

The relativistically covariant form of the Boltzmann equation under relaxation time approximation, initially described here [9], is given by the following

$$p^\alpha \partial_\alpha f(x,p,t) = p^\alpha u_\alpha \frac{f_{eq}(x,p\,;T,u,\mu) - f(x,p,t)}{\tau_{rel}}. \tag{2}$$

where f_{eq} is the appropriate equilibrium distribution function for the species of particle under investigation, in our case, we will consider the Bose-Einstein equilibrium distribution

$$f_{eq}(x,p\,;T,u,\mu) = \left(\exp((p^\alpha u_\alpha - \mu)/T) - 1\right)^{-1}. \tag{3}$$

The spatial dependence of the distribution function is only present in the spatial dependence of the parameters T,u^α and μ.

Energy-momentum conservation is ensured by the requirement that the 4-divergence of the energy-momentum tensor ($T^{\alpha\beta}$) vanishes,

$$\partial_\alpha T^{\alpha\beta} = 0, \tag{4}$$

which under the RTA simplifies to

$$u_\alpha T^{\alpha\beta}_{[f]} = u_\alpha T^{\alpha\beta}_{eq}. \tag{5}$$

If we define the rest frame as the frame in which the net energy flux vanishes, also called the Landau rest frame (LRF), this becomes

$$T^\beta_{[f]\,\alpha}\; u^\alpha = \mathcal{E}_{eq} u^\beta. \tag{6}$$

The (only) positive eigenvalue ($\mathcal{E}_{eq}$) of this eigenvalue equation corresponds to the energy density in the LRF, and the associated eigenvector is the flow velocity as in eq (3). We can, optionally, enforce particle number conservation as well, this would generate the additional requirement,

$$\partial_\alpha J^\alpha = 0, \tag{7}$$

where J^α is the 4-current density. Under the RTA, this condition becomes,

$$u_\alpha J^\alpha = \mathcal{N}_{eq} \tag{8}$$

where $\mathcal{N}_{eq}$ is the local number density in the Landau rest frame. Using these we can determine the local temperature and chemical potential, using

$$\mathcal{E}_{eq} = \frac{3}{\pi^2}\mathrm{Li}_4\left(\exp(\frac{\mu}{T})\right)T^4 \tag{9}$$

$$\mathcal{N}_{eq} = \frac{1}{\pi^2}\mathrm{Li}_3\left(\exp(\frac{\mu}{T})\right)T^3, \tag{10}$$

where Li_n denotes the polylogarithm of order n.

Thus the parameters T, u^α and μ, the temperature, fluid flow velocity, and chemical potential respectively, are constrained and the only free parameter in the model is the relaxation time τ_{rel}.

Even though the Boltzmann equation under the relaxation time approximation is conceptually very simple, the temperature, flow velocity and, possibly, chemical potential are dynamic. The dynamic nature of these parameters have, thus far, rendered an exact analytic solution intractable without assuming a highly symmetric setting.

Thus we will resort to numerical methods to make headway.

3. Numerical Implementation

We want to simulate the evolution of the distribution function on a discretised phase-space. After setting up an initial grid of points, we perform free-streaming and relaxation steps independently. Free-streaming is performed by propagating the grid forward in time using the characteristic lines of the advection equation. In order to perform the relaxation step, we need access to the equilibrium parameters. We use a Gauss-Legendre integration scheme to calculate the energy-momentum tensor which is required to determine the equilibrium parameters. Which we can then use to update the function values using the right-hand side of eq (2).

There are a few significant numerical hurdles that need to be overcome.

Firstly, since we in general, do not want to be restricted to considering highly symmetric initial conditions (as other explorations of the Boltzmann equation tend to assume), we need to solve the Boltzmann equation in the full 6-dimensional phase space. The consequence of this high dimensionality is that the number of points that we a required to simulate to obtain a particular resolution of the distribution function scales rapidly with the size of the system. This is compounded by the fact that the system is rapidly expanding, especially in the longitudinal direction.

Another challenge is that we also have a large range of relevant momentum scales present in the problem. Typical initial conditions would be at least approximately boost-invariant, which would imply that the longitudinal momentum scale on which the distribution function has support grows exponentially with increasing rapidities. Thus for an accurate simulation, we require an appropriate resolution across all the relevant momentum scales.

Furthermore, due to time dilation, parts of the system at different rapidities evolve at possibly very different rates (in the lab frame). The consequence is that some parts of the system will conclude their evolution (*i.e.* thermalise) long before others, which is inefficient.

It will be convenient to choose the following set of coordinates,

$$t = \tau\cosh\eta \tag{11}$$

$$z = \tau\sinh\eta \tag{12}$$

$$k^0 = k_\perp\cosh Y \tag{13}$$

$$k^z = k_\perp\sinh Y. \tag{14}$$

As is convention, z denotes the axis parallel to the beam. We will henceforth refer to these coordinates as eigentime (τ), spacetime-rapidity (η) and momentum-rapidity (Y). In these coordinates, the Boltzmann equation reads,

$$\left(\frac{\partial}{\partial\tau}+\frac{1}{\tau}v_\eta\frac{\partial}{\partial\eta}+\boldsymbol{v}_\perp\frac{\partial}{\partial\boldsymbol{x}_\perp}\right)f=\frac{k_\mu u^\mu}{k_\perp\cosh(Y-\eta)}\frac{f-f_\infty}{\tau_{\rm rel}}. \tag{15}$$

This set of coordinate transformations provide several benefits.

In the case of exact boost-invariance, and zero transverse flow velocity, the eigentime is equivalent to the local proper time, which is the natural timescale on which all parts of the system will evolve. Since physically relevant initial conditions must be approximately boost invariant, and the longitudinal flow velocity, in general, will be much larger than the transverse flow velocity, and thus the eigentime is still a good approximation of the local proper time.

Instead of continually expanding in the longitudinal direction, requiring an ever larger grid to accommodate the distribution function, the long term evolution is such that as $\tau\to\infty$, we have $\eta\to Y$, effectively a compression of the relevant phase space. Thus if the initial condition can be accommodated on the grid, the distribution function can be accommodated for all future times.

Due to locality, each cell can be independently updated, this allows us to develop an algorithm that computes these updates in parallel. This allows us to use Graphical Processing Units (GPUs) which have the ability to provide a large amount of computational power compared to traditional CPUs, as long as the algorithm used can be run in parallel.

4. Results

Using the RTA approximation and assuming boost invariance and cylindrical symmetry, we predict the temperature evolution in both the free-streaming ($\tau_{rel}\to\infty$) and hydrodynamic ($\tau_{rel}\to 0$) limits [7].

In particular in the free-streaming limit

$$\left(\frac{T(\tau)}{T(\tau_0)}\right)=\left(\frac{\tau}{\tau_0}\right)^{-\frac{1}{4}}, \tag{16}$$

and for the hydrodynamic limit,

$$\left(\frac{T(\tau)}{T(\tau_0)}\right)=\left(\frac{\tau}{\tau_0}\right)^{-\frac{1}{3}}. \tag{17}$$

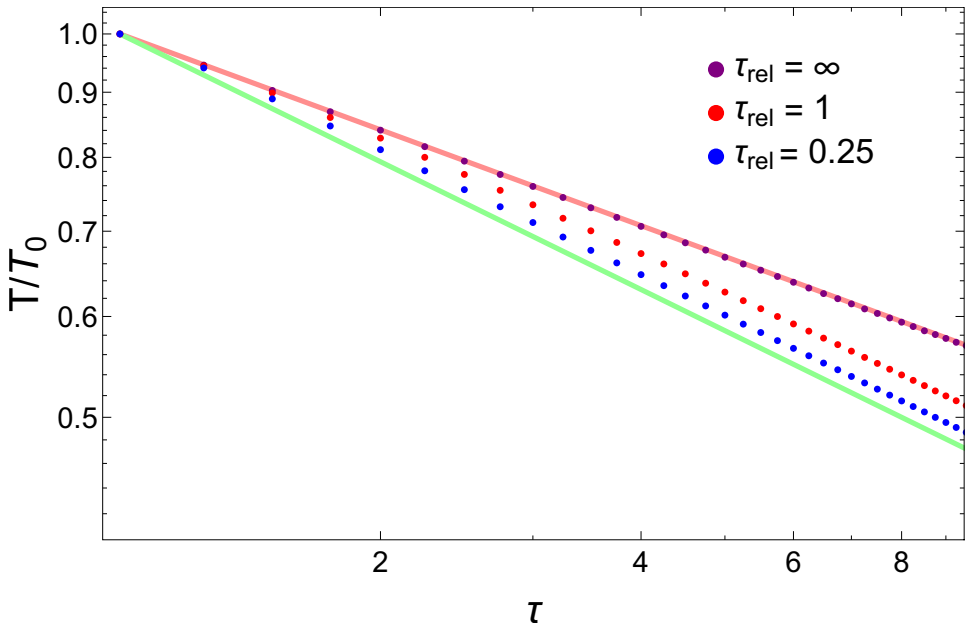

Figure 1: We compare the evolution of the temperature at mid rapidity of various relaxation times (dotted lines) to the hydrodynamic limit (solid green line), and the analytical free-streaming limit (solid red line).

We see that we can not only reproduce the limiting cases but also produce reasonable results for intermediate values of τ_{rel}.

This shows that our simulation is working as intended, and we can begin to investigate more interesting phenomena.

5. Conclusion and Outlook

In this work we have detailed our approach to numerically solving the Boltzmann equation under the relaxation time approximation, and showed preliminary success in reproducing analytical limits.

We hope to begin applying this new numerical tool to experimentally relevant initial conditions to investigate phenomena such as the possibility of Bose-Einstein condensation.

A natural extension of the project would be to consider more physically realistic collision terms. The modular nature of our software would allow us to rapidly prototype these collision terms.

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