Quantum corrections to the kink-antikink potential

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Abstract. We estimate the one loop quantum correction to the kink-antikink potential by computing the vacuum polarization energy as a function of the kink-antikink separation. This energy is calculated from scattering data for fluctuations about the static kink-antikink configuration. To construct a well-defined quantum theory for these fluctuations an additional constraint must be implemented.

1. Introduction

Non-linear field theories may produce classical solutions that have a localized energy density. These solutions are called solitons or solitary waves [1]. In many cases these solitons have particle properties. The most prominent example is the Skrymion as a model for baryons [2, 3][1]. In these identifications the integrated energy density is the mass of the particle. Typically this integral overestimates the actual mass of the particle because quantum corrections are omitted. This is not problematic when investigating properties of a single particle. However, the quantum corrections may become important when comparing configurations with different particle numbers as it occurs, for example, when computing binding energies of compound objects. The leading quantum correction to the soliton energy is the vacuum polarization energy (VPE). The VPE is the renormalized sum of the shifts of the zero point energies of the quantum fluctuations due to their interaction with the background configuration generated by the soliton. Being a quantum field theory calculation a proper renormalization must be applied. In the present study we therefore calculate the VPE of the soliton-antisoliton potential in a renormalizable model with a quartic self-interaction in one time and one space dimensions.

2. VPE and spectral method

The calculation of the VPE from scattering data is by now a well established endeavor in the framework of spectral methods [5]. To apply these methods we call the soliton solution to the field equation $\phi_0(x)$. Then we introduce time dependent fluctuations $\eta(x,t) = \eta_\omega(x)\mathrm{e}^{-i\omega t}$ and linearize the full field equation. This gives rise to a relativistic wave-equation

$$\omega^2\eta_\omega(x) = \left[-\frac{d^2}{dx^2} + U[\phi_0(x)]\right]\eta_\omega(x)\,, \tag{1}$$

where $U[\phi_0(x)]$ is the potential for the fluctuations generated by the background soliton. This wave-equation is a standard problem in potential scattering [6]. The background polarizes the

[1] For a recent review see Ref. [4].

quantum fluctuations $\eta_\omega(x)$ in two aspects. First, it creates bound states with energies ω_j and second, the density of the scattering states is distorted. The scattering states have energies $\omega = \sqrt{k^2+m^2}$ where k is the momentum and m the mass of the fluctuating field. Then the distortion of the density of scattering states is measured by the derivative of the scattering phase shift $\delta(k)$. Weighting the contribution from the scattering states to the VPE accordingly yields

$$E_{\rm vac} = \sum_j \frac{\omega_j}{2} + \sum_{p=\pm} \int_0^\infty \frac{dk}{2\pi} \sqrt{k^2+m^2}\ \frac{d\delta_p(k)}{dk}\Big|_{\rm renorm.}\ , \tag{2}$$

since in one space dimensions there are two scattering channels when the potential is reflection invariant. These two channels are characterized by symmetric ($p=+$) and anti-symmetric ($p=-$) wave-functions when $x \to -x$.

Renormalization is accomplished in two steps. First the divergent contributions to the momentum integral in Eq. (2) are identified from the Born series. They are subtracted under the integral and added back to the VPE as Feynman diagrams. In the second step the divergences of the Feynman diagrams are removed with the help of standard counterterms whose coefficients are universal for a fixed renormalization scheme. For the current problem with boson quantum fluctuations in one space dimension this procedure is quite simple because only the first order tadpole diagram is divergent. This diagram is local and can be fully removed under renormalization. Applying Levinson's theorem to the formal expression and integrating by parts yields the VPE

$$E_{\rm vac} = \frac{1}{2}\sum_j (\omega_j - m) - \sum_{p=\pm} \int_0^\infty \frac{dk}{2\pi} \frac{k}{\sqrt{k^2+m^2}} \left[\delta_p(k) - \delta_p^{(1)}(k)\right] , \tag{3}$$

where $\delta_p^{(1)}(k)$ is the Born approximation to the phase shift in channel p.

3. Kink-antikink configuration

To be specific we consider the ϕ^4 model in one space and one time dimension. This model is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left[\phi^2 - \frac{m^2}{2\lambda}\right]^2 \tag{4}$$

where λ is a coupling constant. The corresponding field equation contains a static solution, the so-called *kink*, $\phi_0(x) = (m/\sqrt{2\lambda})\tanh(mx/2)$. The antikink solution is obtained by spatial reflection. These solutions build the soliton-antisoliton (or kink-antikink) configuration

$$\phi_R(x) = \frac{m}{\sqrt{2\lambda}}\left[\tanh\left(\frac{m}{2}(x-R)\right) - \tanh\left(\frac{m}{2}(x+R)\right) + 1\right] , \tag{5}$$

for which we want to compute the potential as a function of the fixed separation $2R$. There are classical and VPE contributions.

The classical potential is obtained by substituting the parameterization, Eq. (5) into the Lagrangian, integrating over the coordinate x [7] and subtracting twice the kink mass:

$$V_{\rm cl}(R) = \frac{2m^3}{\lambda}\left[Rm + \frac{3}{\tanh(Rm)} - \frac{2+3Rm}{\tanh^2(Rm)} + \frac{2Rm}{\tanh^3(Rm)} - 1\right] . \tag{6}$$

To compute the VPE contribution we need to solve Eq. (1) for $U[\phi_R(x)]$. However, this does not lead to a well-defined quantum theory for the fluctuations as can be seen from the following

argument. When the kink and antikink are widely separated, *i.e.* $R \to \infty$, each possesses the translational zero mode $\omega^2 = 0$ as well as the breather bound state with energy $\omega^2 = 3m^2/4$ [1]. When reducing R, the two zero mode solutions split with one bound state energy squared turning negative. This produces an imaginary energy eigenvalue [9] and must be avoided. We do so by recalling that the distance R is fixed and no fluctuations in this direction should be admitted. This induces the constraint $\int_{-\infty}^{\infty} dx\, \eta_\omega(x) z(x) = 0$ with

$$z(x) = N \frac{d}{dR}\phi_R(x) \qquad \text{and} \qquad N^{-2} = \int_{-\infty}^{\infty} dx\, \left(\frac{d}{dR}\phi_R(x)\right)^2 . \tag{7}$$

This constraint turns the wave-equation into an integro-differential equation that describes a non-local interaction

$$-\eta_\omega''(x) = k^2\eta_\omega(x) - \left(U[\phi_R(x)] - m^2\right)\eta_\omega(x) + z(x)\int_{-\infty}^{\infty} dy\, \left[z(y)\left(U[\phi_R(y)] - m^2\right) - z''(y)\right]\eta_\omega(y)\,. \tag{8}$$

Here primes denote the derivative with respect to the spatial coordinate. Since $z(x)$ vanishes asymptotically, this equation represents a well-defined scattering problem from which phase shifts and their Born approximations can be computed. We also note that the constraint only affects the symmetric ($p = +$) channel. It does not take effect for large separation R where $z(x)$ parameterizes two independent translational vibrations that, by construction, are orthogonal to the remaining fluctuations.

4. Scattering data

To compute the phase shift $\delta_+(k)$ in the symmetric channel from the integro-differential equation (8) we introduce $u(x) = U[\phi_R(x)] - m^2$ and parameterize $\eta_\omega(x) = \mathrm{e}^{\nu_S(x)}\cos[kx + \delta_S(x)]$. This leads to coupled differential equations[2]

$$\begin{aligned} \frac{d\delta_S(x)}{dx} &= -\frac{1}{k}\, c(x)\left[u(x)c(x) - \alpha z(x)\,\mathrm{e}^{-\nu_S(x)}\right] \\ \frac{d\nu_S(x)}{dx} &= -\frac{1}{k}\, s(x)\left[u(x)c(x) - \alpha z(x)\,\mathrm{e}^{-\nu_S(x)}\right] , \end{aligned} \tag{9}$$

where $c(x) = \cos[kx + \delta_S(x)]$ and $s(x) = \sin[kx + \delta_S(x)]$. Furthermore α is a Lagrange multiplier that is iteratively adjusted such that $\int_{-\infty}^{\infty} dx z(x)c(x)\mathrm{e}^{\nu(x)} = 0$. This system of equations is solved with the initial conditions $\delta_S(0) = 0$ and $\nu(0) = 0$ for a given value k. Then $\delta_+(k) = \lim_{x\to\infty}\delta_S(x)$. In the antisymmetric channel we similarly parameterize $\eta_\omega(x) = \mathrm{e}^{\nu_A(x)}\sin[kx + \delta_A(x)]$. Since the constraint is not active in that channel, the wave-equation simplifies to an ordinary differential equation

$$\frac{d\delta_A(x)}{dx} = -\frac{u(x)}{k}\sin^2[kx + \delta_A(x)]\,. \tag{10}$$

The initial condition $\delta_S(0) = 0$ then produces $\delta_-(k) = \lim_{x\to\infty}\delta_A(x)$. The Born approximations $\delta_\pm^{(1)}(k)$ are similarly obtained from the linearized versions of equations (9) and (10) with $\alpha = 0$ since in the renormalization process we need to resemble the full Feynman diagram[3].

As it is the case for the scattering data, *cf.* Eqs. (9) and (10), the bound state wave-functions are either symmetric or antisymmetric under spatial reflection. Of course the discrete sum $\left(\sum_j\right)$

[2] The two functions are related by $c(x)d\nu_S(x)/dx = s(x)d\delta_S(x)/dx$ [8].

[3] Of course, formally we could compute the diagram subject to the constraint, but this would unnecessarily complicate matters.

Table 1. Bound state energies in the symmetric ($\omega_j^{(+)}$) and antisymmetric ($\omega_j^{(-)}$) channels for various values of the separation distance R.

bound state	$R = 0.25$	$R = 0.5$	$R = 0.75$	$R = 1.0$	$R = 3.0$	$R = 4.0$
$\omega_1^{(+)}$	0	0	0	0	0	0
$\omega_2^{(+)}$	—	1.905	1.719	1.592	1.723	1.731
$\omega_1^{(-)}$	1.815	1.350	0.927	0.606	0.012	0.002
$\omega_2^{(-)}$	—	—	1.999	1.955	1.740	1.733

in Eq. (3) comprises these channels. We obtain the bound state energies by first constructing a set of basis states via implementing boundary conditions on the non-interacting solutions ($U \equiv m^2$) at a distance L representing spatial infinity. Within this basis we compute matrix elements of the operator on the right-hand-side of Eq. (1) and find its eigenvalues. Those below m^2 are the bound state energies (squared). Since the bound state wave-functions decay exponentially at large x, the corresponding energies are not sensitive to the particular value of L as long as it is large enough. For the symmetric channel with the constraint we diagonalize the operator from Eq. (1) after sandwiching it between the projector $\mathbb{1} - |z\rangle\langle z|$. Note that this always causes a zero mode to appear for the solution, $|z\rangle$ that is annihilated by the projector.

With these solutions to the constraint wave-equation we compute the VPE from Eq. (3) for distinct values of the distance R. Subtracting twice the VPE of a single (anti)kink yields the leading quantum correction to the kink-antikink potential as a function of R.

5. Results

To produce numerical results we redefine the field and the coordinates such that $m = 2$ and $\lambda = 2$.

In table 1 we list the bound state energies as a function of the kink-antikink separation R that result from the wave-equation (8). A zero mode always appears in the symmetric channel because of the projection. As $R \to \infty$ the lowest bound state, $\omega_1^{(-)}$ in the anti-symmetric channel also turns into a zero mode. In the same limit the second bound state, $\omega_2^{(-)}$ approaches the breather energy at $\sqrt{3}m/2$.

We also note that the resulting phase shift for the constraint scattering problem reproduces Levinson's theorem

$$\delta_+(0) = \left(n_+ - \frac{1}{2}\right)\pi \qquad \text{and} \qquad \delta_-(0) = n_-\pi \tag{11}$$

for scattering in one space dimension [10, 11][4]. Here n_+ and n_- are the number of bound states in the symmetric and anti-symmetric channels, respectively. Note that n_+ also counts the zero mode from the constraint. This statement can readily be verified for $R = 0.5$ for which we plot the phase shifts in figure 1.

In figure 2 we compare our main result for the quantum energy of the kink-antikink system relative to the quantum energy of two independent kinks

$$V_{\rm vac}(R) = E_{\rm vac}(R) - 2E_{\rm vac}^{(\rm kink)} \tag{12}$$

to the classical counterpart from Eq. (6). While the latter is indeed known to be attractive we find that the quantum correction is repulsive. However, in magnitude it is not enough to

[4] A formal proof of the theorem for non-local interactions is presented in Ref. [12].

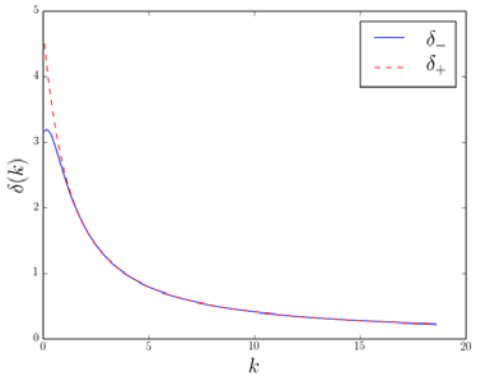

Figure 1. (color online) Phase shift in the symmetric (δ_+) and antisymmetric (δ_-) channels for separation distance $R = 0.5$.

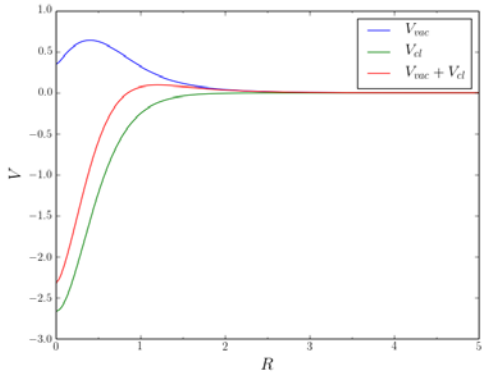

Figure 2. (color online) Quantum correction to kink-antikink potential as a function of the separation R defined in Eq. (12). Also shown is the classical potential from Eq. (6).

turn the whole potential into being repulsive. On the contrary, we find that (in absolute value) the VPE is significantly smaller than the classical potential. This, of course, is consistent with the VPE being a mere correction. Nevertheless, we see from figure 2 that the total potential $V_{\rm cl}(R) + V_{\rm vac}(R)$ produces a small barrier at an intermediate distances. Hence the quantum corrections indeed have the potential to stabilize a classically unstable configuration.

Though we have circumvented the fundamental problem of dealing with imaginary frequencies by introducing a constraint, this leads to an inconsistency that still needs to be resolved. A closer look at figure 2 shows that $V_{\rm vac}$ approximates 0.36 in the limit $R \to 0$. This is actually the trivial configuration which has zero VPE so that $V_{\rm vac} \to -2E_{\rm vac}^{(\rm kink)}$ which in our units is 0.94. In this limit the non-zero result for the VPE originates purely from the constraint.

6. Conclusion and open problems

We have estimated the one-loop quantum correction to the kink-antikink potential. This is a prototype calculation of quantum corrections that require the comparison of soliton energies for different particle numbers. This will eventually shed more light on the predictions for classically stable configurations with large particle numbers such as nuclei [13], for example.

We observe that the quantum correction mitigates the strong attraction seen in the classical kink-antikink potential. Our findings are consistent with the VPE being just a correction to the classical energy. Interestingly enough, though, the quantum correction produces a mild repulsion at intermediate separation suggesting that these corrections stabilize a classically unstable configuration.

We have already mentioned the inconsistency of our VPE result for the trivial configuration $R \to 0$. We note that $\phi_R(x)$ is actually not a solution to the field equations and therefore a static source term in the wave-equation (1) must be added. Currently we investigate how this source term contributes to the kink-antikink potential. There may be both, a direct contribution and an indirect one as a modification of the constraint may turn out inevitable.

Acknowledgments

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