Matrix logarithmic quantum wave equation

Miloslav Znojil[1,2] **and Konstantin G. Zloshchastiev**[1]

[1] Institute of Systems Science, Durban University of Technology, 4000 Durban, South Africa
[2] Nuclear Physics Institute of the CAS, Hlavní 130, 250 68 Řež, Czech Republic

E-mail: znojil@ujf.cas.cz

Abstract. Quantum wave equations with logarithmic nonlinearity are motivated by the conventional (linear) quantum formalism and theory of quantum liquids. The matrix generalization of the logarithmic equation is introduced, which is expected to complement and extend the existing range of physical applications of the conventional ("single-channel") logarithmic wave equation; its properties and possible applications are studied.

1. Introduction

We consider the nonlinear logarithmic Schrödinger equation (LSE)

$$i\partial_t\psi(\vec{x},t) = (-\Delta + V_{LSE})\,\psi(\vec{x},t)\,, \quad V_{LSE} = -b\,\ln|\psi(\vec{x},t)|^2, \tag{1}$$

where Δ is the d-dimensional Laplacian, b is a nonlinear coupling assumed to be a positive constant here; the equation has the wavefunction solutions $\psi \in L^2(\Re^d)$. This equation, as well as its relativistic analogue (which is obtained by replacing the derivative part with the d'Alembert operator), finds numerous applications in extensions of quantum mechanics [1, 2], physics of quantum fields and particles [3, 4, 5, 6, 7], optics and transport or diffusion phenomena, nuclear physics, theory of dissipative systems and quantum information [8, 9], theory of superfluidity [10, 11, 12] and effective models of physical vacuum and classical and quantum gravity [13, 14, 15, 16].

The physical meaning of the solutions ψ and proper interpretation of their dynamics are determined by a phenomenological background of a given application. Still, the generic mathematical features of Eq. (1) (and, first of all, the logarithmic form of its nonlinearity) can be shared. For the purposes of illustration one can select, as a starting point, the context of a linear quantum wave equation. In the Schrödinger picture, the evolution of quantum systems is commonly described by the Schrödinger equation

$$i\partial_t\psi(\vec{x},t) = \hat{H}\psi(\vec{x},t)\,, \qquad \hat{H} = -\Delta + V(\vec{x}), \tag{2}$$

which is linear by construction. In this equation, the choice of the potential $V(\vec{x})$ is usually determined by a number of considerations. Let us consider as an example the exactly solvable (ES) harmonic oscillator $V_{(HO)}(\vec{x}) \sim |\vec{x}|^2$, which has the equidistant, purely vibrational spectrum of the low-lying bound-state energies $E_{(HO)}$ and the localized wavefunctions of the form:

$$\psi_{(HO)}(\vec{x},t) \sim \exp(-iE_{(HO)}t)\,\exp\left(-|\vec{x}|^2/2 + \mathcal{O}(\ln|\vec{x}|)\right), \tag{3}$$

which can be regarded as a natural realization of the above-mentioned Madelung decomposition. The other choices of a potential can be considered on grounds of either the formal relevance of $V(\vec{x}) \neq V_{(HO)}(\vec{x})$ or its capability of describing the variability of the dynamics (by adding perturbations to $V_{(HO)}(\vec{x})$).

Such an asymptotic behavior of wavefunctions may be perceived as related to a certain equivalence-class property of potentials,

$$V_{(initial)}(\vec{x}) = |\vec{x}|^2 + \mathcal{O}(\ln|\vec{x}|) \sim \ -\ln[\psi^*(\vec{x},t)\psi(\vec{x},t)] = V_{(generalized)}(\vec{x}). \tag{4}$$

It is thus not too surprising that the logarithmically nonlinear Schrödinger equation (1) was proposed as a "minimal" generalization of its linear predecessor in quantum theory [1].

2. Logarithmic wave equation in theory of quantum liquids

If one accepts that strongly correlated helium atoms can form a bound state characterized by a single macroscopical wavefunction then the wave equation describing such object cannot be of the Gross-Pitaevskii (GP) type, also known as the cubic Schrödinger (in the relativistic field theory, its analogue would be the quartic scalar field theory ϕ^4). There exist at least two reasons for this. Firstly, the GP approach is a perturbative approach, which takes into account only two-body interactions and neglects anomalous contributions to self-energy, which is a robust approximation for dilute system like cold gases. However, it is unlikely to be sufficient for more dense objects like liquids: according to aforesaid quantum liquid's atoms are delocalized and thus nothing prevents them from getting involved into multiple-body interactions. One example of why multi-body (three and more) interactions are very important for forming bound states of bosons at low temperatures is the Efimov state, which has been experimentally observed in helium. The second issue is that the ground-state wavefunction of the GP BEC model in absence of an external potential trap does not describe a localized object. Instead, the free GP condensate tends to occupy all available volume - as such one needs to apply some kind of a trap, in order to confine the condensate and stabilize the system. This feature is more pertinent to gases than to liquids.

Luckily, there exists another candidate where the above-mentioned issues simply do not occur in the first place. This is nonlinear Bose liquid defined by virtue of the logarithmic Schrödinger equation of the type (1), except that the condensate wavefunction is normalized not to one but to a total amount of condensate particles, see Refs. [14, 10, 11, 12]. The logarithmic Bose liquid has a number of features suitable for our objectives: it implicates not only binary but also multiple-body interactions (when three or more bodies can scatter simultaneously), and its ground state is the so-called gausson – a spherically-symmetric object which is localized and stable even in absence of a trapping potential, with the interior density obeying the Gaussian law

$$\rho = |\psi(\vec{x},t)^2| \sim \exp\left(-|\vec{x}|^2/a^2\right), \tag{5}$$

where a is a characteristic size. Notice that this object is different from the classical droplet because it does not have border in a classical sense: its stability is supported by nonlinear quantum effects in the bulk, rather than by surface tension.

In the work [11], an analytical theory of structure and excitations in superfluid helium was proposed, which elaborates on above-mentioned microscopical aspects and goes beyond the GP approximation. It consists of two nested models which act on different length scales, but are connected via the parametric space: quantities and values of parameters in the long-wavelength model are derived from the short-wavelength part. The short-length model justifies appearance of collective degrees of freedom that can be used for describing the volume elements (fluid parcels) of the quantum liquid. Thus, an intrinsic structure of a fluid parcel is described using

the non-perturbative approach based on the logarithmic wave equation: one can show that the interior density of the element obeys the Gaussian distribution.

The fact that the logarithmic model was so instrumental in describing one of the realistic quantum Bose liquids we know of, gives us a hope that it can be also useful in a theory of physical vacuum. According to the latter, the physical vacuum is described by an essentially non-relativistic superfluid, while the Lorentz symmetry and relativistic gravity emerge in the linearized ("phononic") regime. This can be shown, independently, by means of the modular group approach [13], fluid/gravity correspondence [14], Bogoliubov method [10], and Arnowitt-Deser-Misner formalism [16]. Furthermore, there already exist some experimental data suggesting a connection between the phenomena of superfluidity and gravity [17, 18]. As a matter of fact, the superfluid vacuum paradigm allows us to explore a border of the relativity's applicability range, which is crucial for truly understanding of the spacetime approach, as well as to go beyond it [15].

3. Analytical solutions

In this section we pay our attention to the possible simulations of the interaction between a system and its environment which would be based on the logarithmic type of the interaction's nonlinearity. Let us, therefore, start from the choice of the first nontrivial case and consider solutions of the matrix LSE with $\mathcal{A}$ being a 2×2 matrix. For the sake of simplicity let us also limit our attention to the case of the single spatial dimension replacing vector $\vec{x}$ by scalar x. Then we can write

$$i\partial_t \mathcal{A} + \partial_{xx}\mathcal{A} + b\ln(\mathcal{A}^\dagger\mathcal{A})\mathcal{A} = 0, \tag{6}$$

and we also impose that

$$\int_{x_0}^{x_1} \mathrm{tr}(\mathcal{A}^\dagger\mathcal{A})dx = \mathcal{N}, \tag{7}$$

assuming that the system is confined to the interval of $x \in [x_0, x_1]$. The latter formula comes from the normalization condition $\int d\vec{x}\,\mathrm{tr}(\mathcal{A}^\dagger\mathcal{A}) \equiv \langle\mathcal{A}|\mathcal{A}\rangle = \mathcal{N}$, where integration is taken over a spatial volume taken by the system, $\mathcal{N}$ being a constant usually interpreted as a number of particles inside such a volume, while tr here is the standard matrix-trace operation.

Furthermore, below we consider the cases when we managed to obtain exact analytical solutions of Eq. (6).

3.1. Diagonal case

We shall start our analysis from the simple yet nontrivial case of the two coupled LSEs, $\mathcal{A}$ being a diagonal matrix

$$\mathcal{A} = \begin{pmatrix} \psi_1(x,t) & 0 \\ 0 & \psi_2(x,t) \end{pmatrix}, \tag{8}$$

where $\psi_a(x,t)$ are complex-valued functions. In this model the auxiliary cross-coupling terms are neglected. Considering this case enables us to contemplate, more easily, a pair of uncoupled logarithmic Schrödinger equations in a way which would emphasize also certain parallelism with the concept of the coupled channels in the linear-theory setting. Moreover, in the language of phenomenology, any pair of similar individual equations, linear or nonlinear, may be perceived as mimicking a fully separated evolution of an isolated system $\mathcal{S}_1$ (of our immediate interest) and of its remote and irrelevant, "switched-off" environment $\mathcal{S}_2$.

In the more realistic situations one can only rarely neglect the possible cross-interaction between subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ completely. Still, the most immediate reward of the study of such a mutual interaction usually comes when one assumes that the resulting "perturbation" of the relevant subsystem $\mathcal{S}_1$ remains weak.

The ground-state solution of Eq. (6) can be found exactly. In a rest frame, it has the form of the gausson, i.e., by the Gaussian packet modulated by the de Broglie plane wave,

$$\psi_a(x,t) = C_a \exp\left(-\frac{1}{2} b\, x^2 + \nu_a x - iE_a t\right), \tag{9}$$

where $a = 1$, 2 and where

$$E_a = b(1 - \ln C_a^2) - \nu_a^2 \tag{10}$$

is energy of a wave in the ath channel. Quantities C_a and ν_a are integration constants related, together with b, to the mean and variance of the Gaussian packet.

If one imposes also the normalization condition (7) then one obtains an additional constraint for the integration constants

$$F_1(x_0) - F_1(x_1) + F_2(x_0) - F_2(x_1) = 2\sqrt{\frac{b}{\pi}}\mathcal{N}, \tag{11}$$

where we denoted $F_a(x) = C_a^2 \exp\left(\frac{\nu_a^2}{b}\right) \mathrm{erf}\left(\frac{\nu_a - bx}{\sqrt{b}}\right)$. For instance, when x_1 and x_0 are set to plus and minus infinity then this constraint takes a simple form $\sum_{a=1}^{2} C_a^2 \exp\left(\frac{\nu_a^2}{b}\right) = \sqrt{\frac{b}{\pi}}\mathcal{N}$.

Note that due to the Galilean symmetry of LSE, from solution (9) one can always obtain gausson solutions whose center of mass propagates with velocity v_a, independently for each channel. For instance, one can check that the following function

$$\psi_a(x,t) = C_a \exp\left[-\frac{1}{2}\,(x - v_a t)\,(b(x - v_a t) - 2\nu_a - iv_a)\right], \tag{12}$$

is also a solution of Eqs. (6), (9), provided

$$(v_a/2)^2 + \nu_a^2 = b(1 - \ln C_a^2), \tag{13}$$

v_a being another real-valued integration constant.

3.2. Off-diagonal case

Contrary to the previous case, let us assume that it is the channel-coupling terms which are dominating now. Therefore, one can assume $\mathcal{A}$ being an off-diagonal matrix

$$\mathcal{A} = \begin{pmatrix} 0 & \bar{\psi}_1(x,t) \\ \bar{\psi}_2(x,t) & 0 \end{pmatrix}, \tag{14}$$

where $\bar{\psi}_a(x,t)$ are complex-valued functions.

Still, the ground-state solution of Eq. (6) can be found exactly. In a rest frame, it also has the form of the Gaussian packet modulated by the de Broglie plane wave,

$$\bar{\psi}_a(x,t) = C_a \exp\left(-\frac{1}{2} b\, x^2 + \nu_a x - i\bar{E}_a t\right), \tag{15}$$

where $a = 1$, 2, and

$$\bar{E}_a = \nu_a^2 - b(1 - \ln C_a^2) \tag{16}$$

is energy of a wave, C_a and ν_a are integration constants related, together with b, to the mean and variance of the Gaussian packet. If one imposes also the normalization condition (7) then one obtains an additional constraint for the integration constants, which is identical to Eq. (11).

Comparing the solution (15) with its diagonal analogue (9), one can see an interesting feature: two solutions can be transformed from one another by a simple time inversion

$$\psi_a(x,t) = \bar{\psi}_a(x,-t), \quad E_a = -\bar{E}_a, \tag{17}$$

which can indicate that the diagonal and off-diagonal terms of the matrix $\mathcal{A}$ describe the processes happening in opposite directions of time, or, alternatively, having opposite signs of their energy eigenvalues.

4. Conclusion

The logarithmically nonlinear quantum wave equation, both in its original and matrix forms, is introduced and discussed based on motivations coming from different areas of quantum physics. We also considered some special cases and analytical solutions, for the case of 2×2 matrices. The solutions seem to have a general feature of possessing a Gaussian wave packet shape modulated by the de Broglie plane waves. Besides, the diagonal and off-diagonal components of the matrix turn out to be describing the waves with opposite signs of energy or, alternatively, moving in opposite directions of time. Other types of matrix solutions can be a subject of future studies.

Acknowledgments

Research of K.Z. is supported by the National Research Foundation of South Africa under Grants No. 95965, 98083 and 98892. Research of M.Z. is supported by Grants GAČR 16-22945S and IRP RVO61389005, and his visit to Durban University of Technology was funded by the National Research Foundation of South Africa under Grant No. 98892.

References

[1] Bialynicki-Birula I and Mycielski J 1976 *Annals Phys.* **100** 62
[2] Bialynicki-Birula I and Mycielski J 1979 *Phys. Scripta* **20** 539
[3] Rosen G 1969 *Phys. Rev.* **183** 1186
[4] Dzhunushaliev V and Zloshchastiev K G 2013 *Central Eur. J. Phys.* **11** 325
[5] Gulamov I E, Nugaev E Y and Smolyakov M N 2014 *Phys. Rev.* D **89** 085006
[6] Gulamov I E, Nugaev E Y, Panin A G and Smolyakov M N 2015 *Phys. Rev.* D **92** 045011
[7] Dzhunushaliev V, Makhmudov A and Zloshchastiev K G 2016 *Phys. Rev.* D **94** 096012
[8] Yasue K 1978 *Annals Phys.* **114** 479
[9] Brasher J D 1991 *Int. J. Theor. Phys.* **30** 979
[10] Avdeenkov A V and Zloshchastiev K G 2011 *J. Phys.* B: *At. Mol. Opt. Phys.* **44** 195303
[11] Zloshchastiev K G 2012 *Eur. Phys. J.* B **85** 273
[12] Bouharia B 2015 *Mod. Phys. Lett.* B **29** 1450260.
[13] Zloshchastiev K G 2010 *Grav. Cosmol.* **16** 288
[14] Zloshchastiev K G 2011 *Acta Phys. Polon.* B **42** 261
[15] Zloshchastiev K G 2011 *Phys. Lett.* A **375** 2305
[16] Scott T C, Zhang X, Mann R B and Fee G J 2016 *Phys. Rev.* D **93** 084017
[17] Tajmar M, Plesescu F and Seifert B 2009 *J. Phys. Conf. Ser.* **150** 032101
[18] Tajmar M and Plesescu F 2010 *AIP Conf. Proc.* **1208** 220