A Different Approach to the Perturbation of Astrophysical Fluids

F A M Frescura[1] **and C A Engelbrecht**[2]

[1] School of Physics, U Witwatersrand, 1 Jan Smuts Avenue, Braamfontein, Johannesburg
[2] Department of Physics, U Johannesburg, Kingsway Ave, Auckland Park, Johannesburg

E-mail: fabio.frescura@wits.ac.za

Abstract. We show that Eulerian and Lagrangian perturbations can be interpreted as finite differences that arise from suitably defined differential operators. These operators lead to exact rather than to approximate perturbation relations and equations. The equations obtained are more general than those usually encountered, and are de facto linear. No approximation is thus needed to linearise them. We also explore the possibility of extending this formalism to the description of stellar pulsations of arbitrary amplitude without using power series expansions.

1. Introduction

Perturbation theory considers two flows, called respectively the 'unperturbed' and 'perturbed' flows, and uses these to define two variations, called respectively Eulerian and Lagrangian variations. The Eulerian variation describes changes seen by an observer who is at rest in a given frame of reference. The Lagrangian variation describes changes as seen by an observer who moves with the fluid. Both are finite differences. They therefore make perturbation theory approximate at the outset and intrinsically propagate error into all results.

In this paper, I replace these variations by derivatives and show how these may be used to develop a perturbation theory that is free from error.

2. Eulerian and Lagrangian Variations

Let Q be any property of the flow (density, pressure, etc.). Values in the unperturbed flow are subscripted with zero. Values in the perturbed flow are not subscripted.

The Eulerian variation δQ is the difference $\delta Q = Q - Q_0$. Evaluated at $\vec{x}$ and t, it gives the difference in the two flows at the same space point $\vec{x}$ and time t of the property Q.

Laws that govern material behaviour must be applied to a given element of fluid material. Accordingly, the Lagrangian variation ΔQ in Q is defined as the difference at time t in the value of Q in the two flows of one and the same element of material. Suppose that at time t a given fluid element is at position $\vec{x}$ in the unperturbed flow, and at $\vec{x} + \delta\vec{x}$ in the perturbed flow. The vector $\delta\vec{x}$ is called the Lagrangian displacement of the element. The Lagrangian variation of Q is then defined to be the difference

$$\Delta Q \quad = \quad Q(\vec{x} + \delta\vec{x}, t) - Q_0(\vec{x}, t) \tag{1}$$

Unlike the Eulerian variation, which depends only on $\vec{x}$ and t and so is a field on the spacetime occupied by the fluid, the Lagrangian variation is a function also of $\delta\vec{x}$. It is thus a field on the tangent bundle of the spacetime.

Eulerian and Lagrangian variations are not independent. They are related by the equation

$$\Delta Q \;=\; \delta Q(\vec{x},t) + [Q(\vec{x}+\delta\vec{x},t) - Q(\vec{x},t)] \tag{2}$$

Correct to first order in $\delta\vec{x}$, this relation becomes

$$\Delta Q \;\approx\; \delta Q + \delta\vec{x}\cdot\nabla Q + \cdots \tag{3}$$

Extensive use is made of this approximate relation in standard perturbation theory.

Both Eulerian and Lagrangian variations are finite differences. This leads to a number of difficulties, all of which are avoided in the theory presented below.

3. One parameter families of perturbed flows

Problems introduced by the finite-difference nature of Eulerian and Lagrangian variations are avoided by taking their as $|\delta\vec{x}| \to 0$. This limit cannot be performed if only two flows, perturbed and unperturbed, are considered. We must replace these by a new concept in which the two flows are replaced by a one-parameter family of flows, $\vec{F}(\vec{a},t,\lambda)$, in which one flow corresponds to each value of the family parameter λ. Each of these flows is characterised by its own set of fluid variables (pressure, density, etc.). Thus, each variable will be a function, not only of $\vec{x}$ and t, but also of the parameter λ. A quantity Q will thus be represented by a function $Q(\vec{x},t,\lambda)$. The unperturbed flow is represented by parameter value $\lambda = 0$. Non-zero values of λ correspond to perturbed flows. Small values of λ correspond to small perturbations; larger values, to larger perturbations.

The flow $\vec{F}(\vec{a},t,\lambda)$ is interpreted as follows. Consider the unperturbed flow at time $t = 0$. Each fluid point is at a well defined position $\vec{a}$ in space. We use these coordinates to identify the fluid point at all subsequent times. The values $\vec{a} = (a^1, a^2, a^3)$ are called the Lagrangian coordinates of the material point. As time advances, the fluid point changes its position, which is given at time t by $\vec{F}(\vec{a},t,0)$. As t changes, the pathline traced out by this fluid point is given by the function $\vec{F}(\vec{a},t,0)$ where $\vec{a}$ is held fixed.

In the perturbed flow with parameter value λ, the fluid point with Lagrangian coordinates $\vec{a}$ occupies position $\vec{F}(\vec{a},t,\lambda)$ at time t. As t changes, the function $\vec{F}(\vec{a},t,\lambda)$, where $\vec{a}$ and λ are held fixed, traces out the pathline of this fluid point in the perturbed flow defined by λ.

The set of all possible positions of a given fluid point $\vec{a}$ defines a two dimensional surface whose Gaussian coordinates are t and λ, given by $\vec{F}(\vec{a},t,\lambda)$. The parameters t and λ define a surface grid in which the $\lambda =$ constant curves are the pathlines of the fluid point $\vec{a}$ for different perturbed flows, while the $t =$ constant curves represent the locus of positions assumed by the fluid point $\vec{a}$ at time t as the flow is perturbed or deformed. Interestingly, the function $\vec{F}(\vec{a},t,\lambda)$ with fixed t is also a flow. It is the flow at time t that deforms the unperturbed fluid into its various perturbed configurations at time t. I will call it the *deformation flow*.

I will now relate this formalism to that of standard perturbation theory. The unperturbed flow is defined by the function $\vec{F}(\vec{a},t,0) = \vec{F}_0(\vec{a},t)$ and has fluid properties represented by functions $Q(\vec{x},t,0) = Q_0(\vec{x},t)$. The perturbed flow of standard theory is then a flow corresponding to parameter $\delta\lambda$. It is represented by flow function $\vec{F}(\vec{a},t) = \vec{F}(\vec{x},t,\delta\lambda)$ with properties $Q(\vec{x},t) = Q(\vec{x},t,\delta\lambda)$. More correctly, $\vec{F}$ and Q should be subscripted by $\delta\lambda$, but standard theory does not use subscripts for perturbed quantities.

The Lagrangian displacement of the fluid point $\vec{a}$ is given by

$$\delta\vec{x} = \vec{F}(\vec{a},t,\delta\lambda) - \vec{F}(\vec{a},t,0) \approx \frac{\partial\vec{F}}{\partial\lambda}(\vec{a},t,0)\;\delta\lambda \tag{4}$$

This vector is the finite difference version of the tangent vector to the constant t curves on the deformation surface for the fluid point $\vec{a}$ at parameter value $\lambda = 0$. It is clear that a similar procedure can be implemented for any two adjacent perturbations λ and $\lambda + \delta\lambda$. We thus obtain a field of Lagrangian displacements on each deformation surface, which I will call the *generalised Lagrangian displacements*.

Rather than work with a field of finite differences, it is better to use the vector field from which they are defined. This vector field, which I denote by ξ and call the *deformation field*, is defined by the identity

$$\vec{\xi}(\vec{F}\,(\vec{a},t,\lambda),t,\lambda) = \frac{\partial \vec{F}}{\partial \lambda}(\vec{a},t,\lambda) \tag{5}$$

This identity shows that the constant-t curves on the deformation surface are the integral curves (field lines) of the deformation field. The field $\vec{\xi}$ thus fully defines the 1-parameter family of deformations at each time t.

These observations reduce the problem of perturbation to that of finding the vector field $\vec{\xi}$, and then constructing its integral curves to yield the one parameter family of flows $\vec{F}$. This theory involves no approximations. It is exact from beginning to end. The only approximations that might arise are those needed for numerical solution of the exact equations.

The velocity field of the flow for parameter value λ is defined by

$$\vec{v}\left(\vec{F}(\vec{a},t,\lambda),t,\lambda\right) = \frac{\partial \vec{F}}{\partial t}(\vec{x},t,\lambda) \tag{6}$$

These are the tangent vectors to the constant λ curves on the deformation surface.

The t-λ curves form a Gaussian coordinate grid on the deformation surface. The Lie bracket of the vector fields $\vec{v}$ and $\vec{\xi}$ is thus necessarily zero,

$$[\ \vec{x},\vec{\xi}\ ] = \mathcal{L}_{\vec{v}}\ \vec{\xi} = -\mathcal{L}_{\vec{\xi}}\ \vec{v} = 0 \tag{7}$$

This is a key property of this theory.

4. Eulerian and Lagrangian derivatives

The previous section contains the theoretical infrastructure needed for replacing the finite differences ΔQ and δQ by derivatives. In terms of this formalism, the Eulerian variation is defined by

$$\delta Q \quad = \quad Q(\vec{x},t,\lambda) - Q(\vec{x},t,0) \tag{8}$$

It is clear that this variation can be extended to the entire deformation surface by defining

$$\delta Q_\lambda \quad = \quad Q(\vec{x},t,\lambda+\delta\lambda) - Q(\vec{x},t,\lambda) \tag{9}$$

I will call this the *generalised Eulerian variation*. We can now define an associated derivative by forming the differential quotient

$$\lim_{\lambda\to 0} \frac{Q(\vec{x},t,\lambda+\delta\lambda) - Q(\vec{x},t,\lambda)}{\lambda} = \frac{\partial Q}{\partial \lambda}(\vec{x},t,\lambda) \tag{10}$$

I will call this the *Eulerian derivative* of the 1-parameter family of flows. It is the derivative from which Eulerian variations are defined. Interestingly, it is just the ordinary partial derivative of the 1-parameter family of flows with respect to the family parameter λ.

Similarly, the Lagrangian variation is defined in this formalism by

$$\Delta Q \;=\; Q\left(\vec{F}(\vec{a},t,\lambda),t,\lambda\right) - Q\left(\vec{F}(\vec{a},t,0),t,0\right) \tag{11}$$

This too can be extended to the entire deformation surface. I therefore define the *generalised Lagrangian variation* to be

$$\Delta Q_\lambda \;=\; Q\left(\vec{F}(\vec{a},t,\lambda+\delta\lambda),t,\lambda+\delta\lambda\right) - Q\left(\vec{F}(\vec{a},t,\lambda),t,\lambda\right) \tag{12}$$

The Lagrangian derivative of Q is then defined by forming the limit

$$\frac{DQ}{D\lambda} = \lim_{\lambda\to 0}\frac{\Delta Q_\lambda}{\lambda} \;=\; \lim_{\lambda\to 0}\frac{Q\left(\vec{F}(\vec{a},t,\lambda+\delta\lambda),t,\lambda+\delta\lambda\right) - Q\left(\vec{F}(\vec{a},t,\lambda),t,\lambda\right)}{\lambda} \tag{13}$$

We can evaluate $DQ/D\lambda$ explicitly from the definition using (5) get

$$\frac{DQ}{D\lambda} \;=\; \frac{\partial Q}{\partial\lambda} + \vec{\xi}\cdot\nabla Q \tag{14}$$

If we interpret the function $\vec{F}(\vec{a},t,\lambda$ for fixed t as a flow (the deformation flow), then the Lagrangian derivative is seen to be the material derivative for this flow. It measures the rate of change of Q at fixed time observed by an observer who moves with the fluid as the fluid deforms.

Interestingly, equation (14) relates the Eulerian to the Lagrangian derivative. This result resembles that between Lagrangian and Eulerian variations given in standard theory by $\Delta Q = \delta Q + \delta\vec{x}\cdot\nabla Q$. Note, however, that the standard result is approximate and correct only to first order in $\delta\vec{x}$, whereas result (14) is exact.

5. Some results

Using (7), the material derivatives D/Dt and $D/D\lambda$ can be shown to commute. De facto, the derivatives $\partial/\partial t$ and $\partial/\partial\lambda$ commute. These results are exact and do not need the elaborate proofs found, for example, in Lynden-Bell and Ostriker (1967).

Using these commutation relations and identities (5) and (6), it follows almost trivially that

$$\frac{D\vec{\xi}}{Dt} \;=\; \frac{D\vec{v}}{D\lambda} \tag{15}$$

Result (15) is the differential version of the result usually stated in finite difference form as[1]

$$\delta\vec{v} = \vec{v}(\vec{r},t) - \vec{v}_0(\vec{r}_0,t) = \frac{d(\delta\vec{r})}{dt} \tag{16}$$

Conservation of mass in the perturbed flows is described by the integrated mass equation (Cox, 1980, p 54-55). This result requires considerable effort to prove for a general perturbation using standard theory. The method developed above delivers this result as a trivial consequence of the fact that deformation of the fluid is a flow. Since a fluid element does not change its mass in any flow, it cannot do so in the deformation flow. Hence, we must have

$$\frac{\partial\rho}{\partial\lambda} + \nabla\cdot(\rho\vec{\xi}) \;=\; 0 \tag{17}$$

[1] See, for example, Cox (1980), p 53, eq. (5.20).

or, equivalently,

$$\frac{D}{D\lambda}\ln\rho + \nabla\cdot\vec{\xi} \;=\; 0 \tag{18}$$

The momentum and energy equations are also easily obtained, but with a little more algebra. Essentially, we need to differentiate the flow equations with respect to λ using either the Lagrangian or the Eulerian derivative. This procedure delivers equations for the λ-derivatives of the fluid variables. These derivatives take the place of their Lagrangian and Eulerian variations in ordinary perturbation theory. The equations obtained are de facto linear because of the nature of the operation of differentiation. They therefore do not need to be linearised in a separate step by neglect of higher order terms. In fact, no approximations need be performed to obtain the required perturbations equations. Furthermore, the equations obtained contain all the terms required to describe the effects of large perturbation. As the fluid deformation becomes larger, we expect the forces that act on a fluid element to differ from those that act on it when the distortion is small. The method developed above delivers without additional work the equations of motion of the fluid for all deformations, large or small. If only small perturbations are of interest, the equations of standard perturbation theory can be recovered by setting $\lambda = 0$ and by using the properties of the equilibrium flow to simplify the equations.

To illustrate the procedure, differentiate Euler's equation with respect to λ to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial\vec{v}}{\partial\lambda}\right) + \frac{\partial\vec{v}}{\partial\lambda}\cdot\nabla\vec{v} + \vec{v}\cdot\nabla\left(\frac{\partial\vec{v}}{\partial\lambda}\right) \;=\; \frac{1}{\rho^2}\,\frac{\partial\rho}{\partial\lambda}\,(\nabla p) - \frac{1}{\rho}\,\nabla\left(\frac{\partial p}{\partial\lambda}\right) + \frac{\partial\vec{f}}{\partial\lambda} \tag{19}$$

This is the perturbed momentum equation. the derivative $\partial\vec{v}/\partial\lambda$ plays the role of the Eulerian variation $\vec{v}\,'$ of ordinary perturbation theory, $\partial\rho/\partial\lambda$ that of ρ', $\partial p/\partial\lambda$ that of p', and $\partial\vec{f}/\partial\lambda$ that of $\vec{f}\,'$. To express this equation in terms of the deformation field $\vec{\xi}$, it is better to apply the Lagrangian derivative to Euler's equation, followed by (15). This gives

$$\frac{D^2\vec{\xi}}{Dt^2} \;=\; \frac{1}{\rho^2}\left(\frac{D\rho}{D\lambda}\right)(\nabla p) - \frac{1}{\rho}\,\frac{D}{D\lambda}\,(\nabla p) + \frac{D\vec{f}}{D\lambda} \tag{20}$$

The energy equation for the perturbed fluid is treated similarly. For example, if the energy equation for the unperturbed flow is written in the form

$$\frac{D}{Dt}\ln p \;=\; \Gamma_1\,\frac{D}{Dt}\ln\rho + (\Gamma_3-1)\frac{\rho}{p}\,\left(\varepsilon - \frac{1}{\rho}\,\nabla\cdot\vec{\Phi}\right) \tag{21}$$

where Φ is the heat current density in the fluid and the Γ's are Chandrasekhar's adiabatic indices given by $\Gamma_1 = (d\ln p/d\ln\rho)_s$ and $\Gamma_3 - 1 = (d\ln T/d\ln\rho)_s$, then the corresponding equation for the perturbed flow becomes

$$\begin{aligned}\frac{D}{Dt}\left(\frac{D}{D\lambda}\ln p\right) = \Gamma_1\,\frac{D}{Dt}\left(\frac{D}{D\lambda}\ln\rho\right) + \frac{D\Gamma_1}{D\lambda}\,\frac{D}{Dt}\ln\rho \\ +(\Gamma_3-1)\frac{\rho}{p}\,\frac{D}{Dt}\left[\frac{D}{D\lambda}\left(\varepsilon - \frac{1}{\rho}\,\nabla\cdot\vec{\Phi}\right)\right] + \frac{D\Gamma_3}{D\lambda}\,\frac{\rho}{p}\,\frac{D}{Dt}\left(\varepsilon - \frac{1}{\rho}\,\nabla\cdot\vec{\Phi}\right) \\ +\frac{\Gamma_3-1}{p}\frac{D\rho}{D\lambda}\,\frac{D}{Dt}\left(\varepsilon - \frac{1}{\rho}\,\nabla\cdot\vec{\Phi}\right) - \frac{(\Gamma_3-1)\rho}{p^2}\frac{Dp}{D\lambda}\,\frac{D}{Dt}\left(\varepsilon - \frac{1}{\rho}\,\nabla\cdot\vec{\Phi}\right)\end{aligned} \tag{22}$$

The equations given above are insufficient to determine the Eulerian or Lagrangian derivatives of all of the fluid parameters. Additional constraints must be supplied. These are usually given in the form of a process equation. For example, if the deformation of the fluid is a relatively rapid

process, then the entropy density s of the fluid is conserved during the deformation procedure. This gives rise to adiabatic perturbations. The condition for the perturbation to be adiabatic is easily implemented in the new formalism and takes the form

$$\frac{Ds}{D\lambda} = 0 \tag{23}$$

Non-adiabatic perturbations, in which the perturbation follows some other process equation, are described with the same ease: differentiate the process equation with respect to λ using either the Euler or the Lagrangian derivative. This yields an additional equation to be satisfied by the λ-derivatives of the fluid parameters. Equations that determine the rates of change of properties of fluid material are best formulated in terms of the Lagrangian derivative. They can then be converted to Eulerian derivatives using (14).

After the Eulerian derivatives of the fluid variables have been obtained as functions of $\vec{x}, t$ and λ, these expressions can be integrated with respect to λ to obtain them as functions of $\vec{x}, t$ and λ. This provides a full solution of the perturbation problem without approximation. Of course, it is unlikely that the equations obtained will be integrable by analytical methods. Approximation procedures may need to be used for the integration.

6. Conclusion

The principal advantage of the method described above over the standard method is that the equations obtained are themselves free of approximation, unlike their counterparts in the standard theory. Approximations need only be introduced, if necessary, when solving the final equations.

An added advantage of the above method is clarity. The conceptual framework outlined in Sections 3 and 4 is sufficiently well defined to bypass completely and discussion about which flow, perturbed or unperturbed, should be used for defining the material derivatives of interest. It also provides clear proofs of the central results on which the theory is based.

References

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Lynden-Bell, D., and Ostriker, J.P., 1967, MNRAS, **136**, 293.