Small $(q-1)$ expansion of the Tsallis distribution and study of particle spectra at LHC

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Abstract. The fact that in most of the fits to the hadronic spectra the Tsallis q parameter is close to 1, enables us to expand the Tsallis distribution in the Taylor's series of $(q-1)$. Tsallis thermodynamics has been studied with the help of this expansion and the particle spectra have been fitted. Also, the effect of the inclusion of the collective flow in Tsallis distribution has been investigated.

1. Introduction

The transverse momentum distributions of hadrons at high energies is very ofter described by the Tsallis distribution [1]. The PHENIX and STAR collaborations [2, 3] at the Relativistic Heavy Ion Collider (RHIC) at BNL and by the ALICE, ATLAS and CMS collaborations [4, 5, 6, 7, 8] at the Large Hadron Collider (LHC) at CERN has made extensive use of this non-extensive distribution. Tsallis distribution has been very successful in explaining the experimental transverse momentum distribution, longitudinal momentum fraction distribution as well as rapidity distribution of hadrons off the e^+e^- as well as $p-p$ collisions [9, 10, 11, 12, 13, 14, 15]. Here we use a Thermodynamically consistent form of the Tsallis distribution, described in detail in [16, 17].

The relevant thermodynamic quantities can be written as integrals over the following distribution function:

$$f = \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{1}{q-1}}. \quad (1)$$

Although the q and the T parameter were shown to be consistent for all the particle species [15, 17], the studies [18, 19] leave ample room to scrutinize this conclusion.

2. Review of Tsallis thermodynamics and its application to high-energy Physics

The entropy density, s, particle number density, n, energy density, ϵ, and the pressure, P in Tsallis thermodynamics given by [17],

$$s = -g \int \frac{d^3p}{(2\pi)^3} \left[f^q \ln_q f - f\right], \quad (2)$$

$$
\begin{aligned}
n &= g\int \frac{d^3p}{(2\pi)^3} f^q, & (3)\\
\epsilon &= g\int \frac{d^3p}{(2\pi)^3}\; E\; f^q, & (4)\\
P &= g\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}\; f^q. & (5)
\end{aligned}
$$

where g is the degeneracy factor.

The function appearing in Eq. 2 is often referred to as q-logarithm and is defined by

$$\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$$

The first and second laws of thermodynamics lead to the following two differential relations:

$$d\epsilon = T\; ds + \mu\; dn, \tag{6}$$
$$dP = s\; dT + n\; d\mu. \tag{7}$$

where, $s = S/V$ and $n = N/V$ are the entropy and particle number densities, respectively (V is the volume).

It is seen that if we use f^q in stead of f to define the thermodynamic variables, the above equations satisfy the thermodynamic consistency conditions which require that the following relations to be satisfied:

$$T = \left.\frac{\partial\epsilon}{\partial s}\right|_n, \tag{8}$$
$$\mu = \left.\frac{\partial\epsilon}{\partial n}\right|_s, \tag{9}$$
$$n = \left.\frac{\partial P}{\partial\mu}\right|_T, \tag{10}$$
$$s = \left.\frac{\partial P}{\partial T}\right|_\mu. \tag{11}$$

Eq. 8, in particular, shows that the variable T appearing in Eq. 1 can indeed be identified as a thermodynamic temperature and is more than just another parameter. It is straightforward to show that these relations are indeed satisfied [17].

Based on the above expressions the particle distribution can be rewritten, using variables appropriate for high-energy physics as

$$\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \left(1 + (q-1)\frac{m_T}{T}\right)^{-\frac{q}{q-1}} \tag{12}$$

at chemical potential $\mu = 0$ and rapidity $y = 0$.

3. Taylors expansion of Tsallis distribution and Thermodynamic variables

In all fits to the experimental data (transverse momentum spectra, for example) the value of q is very close to 1 and hence, for analytical simplicity, we can expand the Tsallis distribution in the Taylor's series of $(q-1)$. The Taylor's expansion is given by (up to oder $(q-1)^2$) [20]:

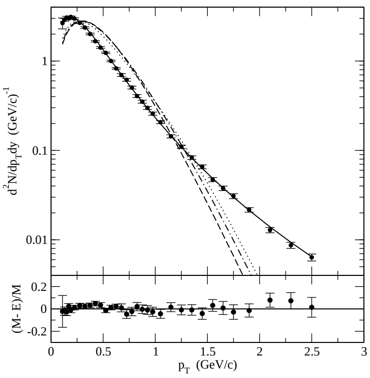

Figure 1. Fits to the normalized differential yields of π^+ as measured by the ALICE collaboration in $p-p$ collisions at $\sqrt{s} = 0.9$ TeV [5] fitted with the Tsallis (solid line) and Boltzmann distributions (dashed line). Also shown are fits with the Tsallis distribution keeping terms to first (dash-dotted line) and second order in $(q-1)$ (dotted line). The lower part of the figure shows the difference between model (M) and experiment (E) normalized to the model (M) values.

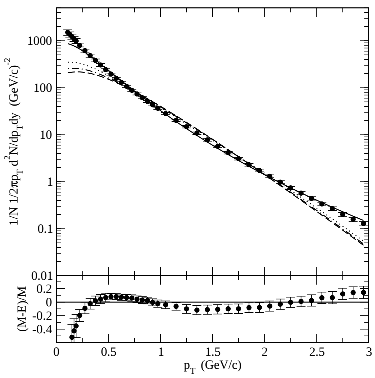

Figure 2. Fits to the normalized differential π^- yields as measured by the ALICE collaboration in $(0-5)\%$ Pb-Pb collisions at $\sqrt{s_{\rm NN}} = 2.76$ TeV [6] fitted with the Tsallis (solid line) and Boltzmann distributions (dashed line). Also shown are fits with the Tsallis distribution keeping terms to first (dash-dotted line) and second order in $(q-1)$ (dotted line). The lower part of the figure shows the difference between model (M) and experiment (E) normalized to the model (M) values.

$$\left[1+(q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} \simeq \mathrm{e}^{-\frac{E-\mu}{T}}\left\{1+(q-1)\frac{1}{2}\frac{E-\mu}{T}\left(-2+\frac{E-\mu}{T}\right)\right.$$
$$\left. + \quad \frac{(q-1)^2}{2!}\frac{1}{12}\left[\frac{E-\mu}{T}\right]^2\left[24-20\frac{E-\mu}{T}+3\left(\frac{E-\mu}{T}\right)^2\right]\right\} \tag{13}$$

The thermodynamic variables keeping up to $O(q-1)$ of the expansion in 13 are calculated below:

3.1. Number density

The particle density in Tsallis thermodynamics is given to first order in $(q-1)$ by the following expression:

$$n \approx n^B + (q-1)n^1 \tag{14}$$

where n^B is the standard Boltzmann result for the particle density:

$$n^B = \frac{g}{2\pi^2} e^{\frac{\mu}{T}} T^3 a^2 K_2(a), \tag{15}$$

with $a \equiv m/T$, and the first order expression in $q-1$ is given by

$$\begin{aligned} n^1 &= \frac{g e^{\frac{\mu}{T}} T^3}{4\pi^2} \Big[-6a^2 K_2(a) - 2a^3 K_1(a) \\ &-4a^2 b K_2(a) + 3a^3 K_3(a) + a^4 K_2(a) + a^2 b^2 K_2(a) \\ &-2a^3 b K_1(a) \Big] . \end{aligned} \tag{16}$$

where k_n are the modified Bessel's function of second kind.

3.2. Energy density

$$\epsilon \approx \epsilon^B + (q-1)\epsilon^1 \tag{17}$$

$$\epsilon^B = \frac{g e^{\frac{\mu}{T}} T^4}{2\pi^2} (3a^2 K_2(a) + a^3 K_1(a)) \tag{18}$$

$$\begin{aligned} \epsilon^1 &= \frac{g e^{\frac{\mu}{T}} T^4}{4\pi^2} \Big[9a^3 K_3(a) + 4a^4 K_2(a) + a^5 K_1(a) \\ &+2b \Big(3a^2 K_2(a) + a^3 K_1(a) - 3a^3 K_3(a) + a^4 K_2(a) \Big) \\ &b^2 \Big(3a^2 K_2(a) + a^3 K_1(a) \Big) \Big] \end{aligned} \tag{19}$$

3.3. Pressure

Finally, the pressure is given by

$$P \approx P^B + (q-1)P^1 \tag{20}$$

$$P^B = \frac{g e^{\frac{\mu}{T}} T^4 a^2 K_2(a)}{2\pi^2} \tag{21}$$

$$\begin{aligned} P^1 &= \frac{g e^{\frac{\mu}{T}} T^4}{4\pi^2} \Big[a^4 K_2(a) + 3a^3 K_3(a) \\ &-2a^3 b K_3(a) + a^2 b^2 K_2(a) + 2a^2 b K_2(a) \Big] \end{aligned} \tag{22}$$

4. Description of the experimental data

In Fig. 1 we show fits to the transverse momentum distribution of π^+ in p-p collisions at 900 GeV. For the Tsallis distribution (solid line) the parameters $T = 70.8$ MeV, $q = 1.1474$. The volume parameter V corresponds to a spherical radius of 4.81 fm. For the Boltzmann distribution (dashed line) the parameters $T = 150.2$ MeV, while the radius used to determine the volume was fixed at a value of 2.65 fm. For the fit using the Boltzmann distribution and the first order term in $(q-1)$ (dashed-dotted line) the values are $T = 138.4$ MeV, $q = 1.035$ while the radius is given by 2.80 fm. In the last case corresponding to Boltzmann plus first and second orders in $(q-1)$ (dotted line) one has $T = 121.2$ MeV, $q = 1.065$ and a radius of 3.09 fm. As is well-known and evident, the fit using the Tsallis distribution is very good.

In Fig. 2 we show fits to the normalized differential π^- yields in $(0-5)\%$ Pb-Pb collisions at $\sqrt{s_{\rm NN}} = 2.76$ TeV as measured by the ALICE collaboration [6] with the Tsallis (solid line) and Boltzmann distributions (dashed line). Fits with the Tsallis distribution keeping terms to first order (dash-dotted line) and second order in $(q-1)$ (dotted line) are also shown. The lower part of the figure shows the difference between the Tsallis distribution (M) and experiment (E). It is clear that the best fit is achieved with the full Tsallis distribution, whereas, using the Boltzmann distribution the description is not good. Successive corrections in $(q-1)$ improve the description. There is a clear deviation at very low transverse momentum (below 0.5 GeV) and also at higher values above 2.75 GeV.

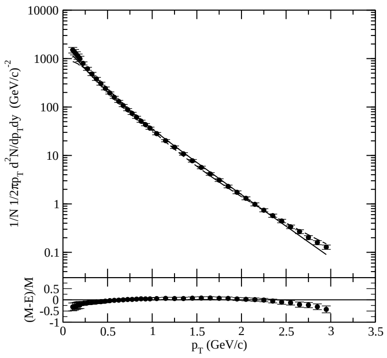

Figure 3. Fits to the normalized differential π^- yields as measured by the ALICE collaboration in $(0-5)\%$ Pb-Pb collisions at $\sqrt{s_{\rm NN}} = 2.76$ TeV [6]. The fit with the Tsallis distribution including flow keeping terms to first order in $(q-1)$ (dashed line). The flow velocity is fixed at $v = 0.609$, with $T = 146$ MeV, $q = 1.030$ and the radius of the volume is $R = 29.8$ fm. The solid line is the Tsallis distribution without flow as given in Fig. 2. The lower part of the figure shows the difference between model (M), i.e. Tsallis with flow up to first order in $(q-1)$, and experiment (E) normalized to the model (M) values.

5. Inclusion of Flow to First Order in $(q-1)$

With a view to see whether the inclusion of flow could improve the description of the transverse momentum distributions obtained in Pb-Pb collisions, we have included a constant flow velocity, v. For space-like freeze-out surfaces, the invariant yield is given by the following formula.

$$\begin{aligned}\frac{1}{p_T}\frac{dN}{dp_Tdy} &= \frac{gV}{(2\pi)^2}\Big\{2T[rI_0(s)K_1(r) - sI_1(s)K_0(r)] - (q-1)Tr^2I_0(s)[K_0(r)+K_2(r)] \\ &+4(q-1)\ TrsI_1(s)K_1(r) - (q-1)Ts^2K_0(r)[I_0(s)+I_2(s)] \\ &+\frac{(q-1)}{4}Tr^3I_0(s)[K_3(r)+3K_1(r)] - \frac{3(q-1)}{2}Tr^2s[K_2(r)+K_0(r)]I_1(s) \\ &+\frac{3(q-1)}{2}Ts^2r[I_0(s)+I_2(s)]K_1(r) - \frac{(q-1)}{4}Ts^3[I_3(s)+3I_1(s)]K_0(r)\Big\} \qquad (23)\end{aligned}$$

where

$$r \equiv \frac{\gamma m_T}{T} \tag{24}$$
$$s \equiv \frac{\gamma v p_T}{T} \tag{25}$$

$I_n(s)$ and $K_n(r)$ are the modified Bessel functions of the first and second kind. In this formula, the freeze-out surface has been considered to be space-like and so the integration over the freeze-out surface turns out to be trivial. For a more detailed treatment of the freeze-out surface in this context, the readers are referred to Ref. [21].

The comparison between model and experiment is quite good with notable deviations at small values of the transverse momentum p_T and again above values of 2.5 GeV (see Fig. 3). These could easily be attributed to the coarse way of treating transverse flow. More detailed investigations have been carried out in [18].

6. Summary and Conclusion

Fits to the particle spectra using the Taylor's expansion are limited to certain ranges of transverse momentum. This may be attributed to the fact that there are certain constraints on the values q, E and T take for these expansions to be valid. Hence, we need to carry out more rigorous treatments to find out exact analytical forms of the Tsallis thermodynamic variables.

Acknowledgments

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