Does quantum discord captures quantum correlations?

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Abstract. Quantum discord is interpreted as a measure of quantum correlations in a composite quantum system. It is defined as the difference of two classically identical expressions for mutual information. In this article we show that the quantum discord can be increased by local operations. Thus, there are classically correlated quantum states with non-zero quantum discord which shows that non-vanishing quantum discord is not a sufficient condition for quantum correlations.

1. Introduction

The characterization of quantum and classical correlations plays an important role in the foundations of quantum theory and has implications for quantum computing. Quantum algorithms require non-classical correlations to improve the efficiency compared to their classical counterpart. Violation of Bell's inequality is a necessary condition for the correlations to be quantum. Entangled states are necessarily non-classically correlated since one can not prepare an entangled state locally. However, a separable state, i.e, not an entangled state, can be prepared by local operations and classical communication. One might ask the question whether entanglement and non-classical correlations are synonymous.

Recent studies have shown that there is more in the quantum correlations than just entanglement [1], namely, quantum discord. In [2] Knill and Laflamme showed that sometimes a highly mixed state with almost no entanglement can perform a task exponentially faster than any classical algorithm. A vast literature is available on the advantage of non-zero discord states in quantum algorithms [3, 4, 5, 6].

In this article we address the problem whether all non-zero discord states contain non-classical correlations. For that purpose we use the geometric measure of quantum discord and the normal form of completely positive trace-preserving maps. We show that a class of non-zero discord states can be prepared from zero discord states by local operations which preserves the trace. Since local trace-preserving operations cannot increase non-classical correlations[1], we argue that such states do not contain any non-classical correlation even though they have non-zero discord.

The article is organised as follows: we start with the definition of quantum discord in Sec. **2**. In Sec. **3** we introduce the geometric measure of quantum discord. We discuss the criterion for the discord to be zero in a given state. This criterion is used to show that local operations can

[1] Correlations in general cannot be increased by local operations which are trace preserving, i.e, deterministic.

increase the discord in Sec. **4**. We conclude the article by a brief discussion on the implications of the results of Sec. **4**.

2. Definition of quantum discord

The Shannon entropy $H(X)$ quantifies the ignorance of a classical random variable X. It is defined as $H(X) = -\sum_x p_x \log_2 p_x$, where $x \in X$ and p_x is the probability of the event x. The quantum counterpart of the Shannon entropy is the von-Neumann entropy $S(\rho)$ for a density matrix ρ defined as $S(\rho) = -\mathrm{tr}\rho \log_2 \rho$. The correlation between two random variables is captured by the mutual information,

$$\mathcal{I}(X:Y) = H(X) + H(Y) - H(X,Y), \tag{1}$$

where $H(X,Y) = -\sum_{x,y} p_{xy} \log_2 p_{xy}$ is the joint information in the two random variables X and Y and p_{xy} is the joint probability of the event x and y. We can replace the joint information by the conditional entropy $H(X|Y)$ by using simple rules in probability theory as:

$$H(X,Y) = H(Y) + H(X|Y), \tag{2}$$

where $H(X|Y) = \sum_y p_y H(X|Y=y)$ represents the information in the variable X when we condition on the output of Y. Substituting Eq. (2) into Eq. (1) results in

$$\mathcal{I}(X:Y) = H(X) - H(X|Y) =: \mathcal{J}(X:Y). \tag{3}$$

This shows that the expression in Eq. (1) and in Eq. (3) are identical in classical probability theory. In the quantum picture the two expressions reads:

$$\mathcal{I}(\rho_{AB}) = S(A) + S(B) - S(AB), \tag{4}$$

$$\mathcal{J}(\rho_{AB}) = S(A) - S(A|B), \tag{5}$$

where $S(A) = -\mathrm{tr}[\rho_A \log_2 \rho_A]$ with $\rho_A = \mathrm{tr}_B \rho_{AB}$.

The conditional entropy $S(A|B)$ is a more involved concept. This is the entropy in the subsystem A conditioned on the measurement outcomes of B. Consider a projective measurement in the orthonormal basis $\{|\psi_i\rangle\}$ in B. For the j-th measurement outcome of B the state of the subsystem A is given by:

$$\rho_A^{(j)} = \frac{\mathrm{tr}_B\left(\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|(\rho_{AB})\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|\right)}{\mathrm{tr}\left(\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|(\rho_{AB})\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|\right)}. \tag{6}$$

The probability of such an event reads

$$p_j = \mathrm{tr}\left(\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|\rho_{AB}\mathbb{I}\otimes|\psi_j\rangle\langle\psi_j|\right). \tag{7}$$

Then the conditional entropy can be written as:

$$S(A|B) = \sum_j p_j S(\rho_A^{(j)}). \tag{8}$$

It is apparent that the conditional entropy $S(A|B)$ depends on the measurement we perform on the subsystem B. Therefore, it is different for different measurement basis in B. In [1] Ollivier and Zurek have shown that the two classically identical expressions for mutual information can be very different in the quantum picture and the difference between them captures the *quantumness* of the correlations.

Quantum discord (with respect to A) is defined as the difference between $\mathcal{I}$ and $\mathcal{J}$ maximized over all the measurements $\{\Pi_i\}$ over subsystem B. It can be expressed by:

$$D_A(\rho_{AB}) = \max_{\{\Pi_i\}}(S(B) - S(AB) - S(A|B)), \tag{9}$$

Interestingly, for pure states entanglement is equivalent to quantum discord. But in the case of mixed states, there are some states which are separable with non-vanishing quantum discord. For example, all the separable Werner states $|w\rangle$ of the form:

$$|w\rangle = \frac{1-p}{4}\mathbb{I} + p|\phi^+\rangle\langle\phi^+|,$$

where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $p \leq 1/3$, have non-zero quantum discord.

Another interesting point to note is that in general the left quantum discord D_A and the right discord D_B are not equal. The states ρ for which both, the left discord and the right discord, are zero, is often referred to as *completely classically correlated* states [7].

3. Geometric measure of quantum discord

Like the separable states, the states with vanishing discord are of great importance. It was shown in [8] that a state σ has zero discord (with respect to B) if, and only if, there exists a projective measurement $\{\Pi_j = |\psi_j\rangle\langle\psi_j|\}$ such that

$$\sum_j (\mathbb{I} \otimes \Pi_j)\sigma(\mathbb{I} \otimes \Pi_j) = \sigma. \tag{10}$$

Alternatively, all the states of the form $\sigma = \sum_j p_k \sigma_A^{(j)} \otimes \Pi_j$ have zero discord with respect to B.

Eq. (10) serves as a criterion to check whether a given state is of zero discord state or not. Next, we can ask what is the amount of discord a given state possesses. To answer that we consider the geometric measure of discord given in [7] for two-qubit systems. For the geometric measure of discord we calculate the distance between a given density matrix ρ and the nearest zero-discord state σ. To calculate the geometric measure of discord, let us have a look at the Bloch representation of a two-qubit density matrix ρ. In the Bloch representation it can be written as:

$$\rho = \frac{1}{4}\left(\mathbb{I} + \boldsymbol{r.\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{s.\sigma} + \sum_{k,l=1}^{3} t_{kl}\sigma_k \otimes \sigma_l\right), \tag{11}$$

where $\boldsymbol{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, $\boldsymbol{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$ and $t_{k,l}$ are real numbers. We can represent the density matrix by another 4×4 matrix M_ρ given by:

$$M_\rho = \frac{1}{4}\left(\begin{array}{c|c} 1 & \boldsymbol{s}^T \\ \hline \boldsymbol{r} & t \end{array}\right). \tag{12}$$

Here $\boldsymbol{s}^T$ is the transpose of vector $\boldsymbol{s}$.

In this new notation the left discord D_A and the right discord D_B can be written as [7]:

$$D_A = \mathrm{tr}(\boldsymbol{r}\boldsymbol{r}^T + tt^T) - l_{max}, \tag{13}$$

$$D_B = \mathrm{tr}(\boldsymbol{s}\boldsymbol{s}^T + t^T t) - e_{max}, \tag{14}$$

where l_{max} is the largest eigenvalue of the matrix $\boldsymbol{r}\boldsymbol{r}^T + tt^T$ and e_{max} is the largest eigenvalue of the matrix $\boldsymbol{s}\boldsymbol{s}^T + t^T t$.

It is easy to see from Eq. (13) and Eq. (14) that if the matrix $L = \boldsymbol{r}\boldsymbol{r}^T + tt^T$ and the matrix $R = \boldsymbol{s}\boldsymbol{s}^T + t^T t$ are rank one matrices, then that state ρ is completely classically correlated.

4. Does non-zero discord imply quantum correlations?

In this section we are addressing the question whether if we have a state ρ with non-zero discord, does that mean the state has true quantum correlations? Quantum correlation are those correlations which are not classical correlations (i.e, captured by mutual information). Moreover, like any correlations they can not be increased by local trace preserving operations. Therefore, if a state ρ with non-zero discord can be prepared by applying local trace preserving operations from a zero-discord quantum state $\tilde{\rho}$, then the correlations in the state ρ are not quantum. Before we discuss the cases where we can derive a non-zero discord state from a zero discord state we need to introduce the *normal form* of completely positive maps.

4.1. Qubit maps: normal form

Consider a completely positive single-qubit map $\Lambda : \mathcal{H}_2 \to \mathcal{H}_2$. Choosing normalized Pauli matrices as operator basis (with $\sigma_0 = \mathbb{I}$) we can represent Λ as a 4×4 matrix M as [9]:

$$M_{\mu\nu} = \mathrm{tr}[\sigma_\mu \Lambda(\sigma_\nu)]/2, \tag{15}$$

where $\mu, \nu = \{0, 1, 2, 3\}$. If Λ is a hermiticity preserving map then the matrix M is real and if Λ is trace preserving then $M_{1\nu} = [1, 0, 0, 0]$. Therefore, the most general trace preserving single qubit map can be written as:

$$M = \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{x} & \lambda \end{array} \right), \tag{16}$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ and λ is a 3×3 real matrix.

The action of a single-qubit channel on one side of two-partite system with a two-qubit density operator ρ_{AB} reads:

$$\begin{aligned} \Lambda \otimes \mathbb{I}(\rho_{AB}) &= M_\Lambda M_{\rho_{AB}} && (17) \\ &= \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{x} & \lambda \end{array} \right) \frac{1}{4} \left(\begin{array}{c|c} 1 & \boldsymbol{s}^T \\ \hline \boldsymbol{r} & t \end{array} \right) && (18) \\ &= \frac{1}{4} \left(\begin{array}{c|c} 1 & \boldsymbol{s}^T \\ \hline \boldsymbol{x} + \lambda \boldsymbol{r} & \boldsymbol{x}\boldsymbol{s}^T + \lambda t \end{array} \right) && (19) \end{aligned}$$

and

$$\begin{aligned} \mathbb{I} \otimes \Lambda(\rho_{AB}) &= M_{\rho_{AB}} M_\Lambda^T && (20) \\ &= \frac{1}{4} \left(\begin{array}{c|c} 1 & \boldsymbol{s}^T \\ \hline \boldsymbol{r} & t \end{array} \right) \left(\begin{array}{c|c} 1 & \boldsymbol{x}^T \\ \hline 0 & \lambda^T \end{array} \right) && (21) \\ &= \frac{1}{4} \left(\begin{array}{c|c} 1 & \boldsymbol{x}^T + \boldsymbol{s}^T \lambda^T \\ \hline \boldsymbol{r} & \boldsymbol{r}\boldsymbol{x}^T + t\lambda^T \end{array} \right). && (22) \end{aligned}$$

With this knowledge of maps and their action of the two-qubit density matrices we now move on to our main result.

4.2. Local operations can increase the discord

In this subsection we will study a class of states with zero discord. We will show that the discord in such states can be increased by almost any arbitrary local completely positive map. Consider the density matrix of the form (in Bloch representation):

$$M_\rho \in \left\{ \frac{1}{4} \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{r} & \alpha \boldsymbol{r}\boldsymbol{r}^t \end{array} \right) \right\}, \tag{23}$$

where $\boldsymbol{r} \in \mathbb{R}^3$ and $\alpha \leq 1$ is an arbitrary real number. The property of such states is that all of these states are zero discord state. The action of the map Λ on the first qubit results in:

$$\Lambda \otimes \mathbb{I}(M_\rho) = \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{x} & \lambda \end{array}\right) \frac{1}{4} \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{r} & \alpha \boldsymbol{r}\boldsymbol{r}^T \end{array}\right) \tag{24}$$

$$= \frac{1}{4} \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{x} + \lambda \boldsymbol{r} & \alpha\lambda \boldsymbol{r}\boldsymbol{r}^T \end{array}\right). \tag{25}$$

It is clear from Eq. (25) that the matrix $L = (\boldsymbol{x} + \lambda\boldsymbol{r})(\boldsymbol{x} + \lambda\boldsymbol{r})^T + \alpha^2\lambda\boldsymbol{r}\boldsymbol{r}^T\boldsymbol{r}\boldsymbol{r}^T\lambda$ is not a rank one matrix and hence has non-zero discord. The application of the same map Λ on the second qubit results in:

$$\mathbb{I} \otimes \Lambda(M_\rho) = \frac{1}{4} \left(\begin{array}{c|c} 1 & 0 \\ \hline \boldsymbol{r} & \alpha \boldsymbol{r}\boldsymbol{r}^T \end{array}\right) \left(\begin{array}{c|c} 1 & \boldsymbol{x}^T \\ \hline 0 & \lambda^T \end{array}\right) \tag{26}$$

$$= \frac{1}{4} \left(\begin{array}{c|c} 1 & \boldsymbol{x}^T \\ \hline \boldsymbol{r} & \boldsymbol{r}\boldsymbol{x}^T + \alpha \boldsymbol{r}\boldsymbol{r}^T\lambda^T \end{array}\right). \tag{27}$$

Here the left discord is still zero but the right discord has become non-zero. Thus we have proved that not all the non-zero discord states have quantum correlations.

5. Discussion

Quantum discord was supposed to capture the non-quantum nature of the correlations. But it can be altered by local operations, therefore, calling quantum discord a signature of quantum correlations is unjustified. However, there is no doubt that the non-zero discord for a quantum state is purely a manifestation of the quantum nature of the system. The question remains what is the amount of quantum correlation in a general quantum state with non-zero discord.

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