Magnetic vector charges in the mystery of a circular current's *pair* of distinct Cartesian elemental magnetic dipoles

M Chirwa

Physics Department, Faculty of Science, Engineering and Technology, Walter Sisulu University, P/Bag X1, Nelson Mandela Drive, Mthatha 5117, South Africa.

E-mail: mmchirwa@wsu.ac.za

Abstract. A circular electric current, perpendicularly bisected by a *field plane*, was modelled as a continuum of pairs of distinct Cartesian component elemental magnetic *vector* charges normal and parallel to the field plane. The Cartesian component elemental magnetic *vector* charges normal to the field plane pair up into Cartesian elemental magnetic dipoles with intra-dipolar displacements parallel to the field plane. These dipoles generate the overall magnetic vector potential. The Cartesian elemental magnetic *vector* charges parallel to the field plane form Cartesian elemental magnetic dipoles (with intra-dipolar displacements perpendicular to the field plane) which individually and collectively contribute nothing to the magnetic vector potential. Each continuum of these two sets of Cartesian elemental magnetic dipoles independently yields the traditionally renowned "magnetic dipolar moment of a circular current". However, together their distinct magnetic fields and their distinct magnetic torques respectively constitute the circular current's overall magnetic field and magnetic torque. These results reconcile only if the magnetic dipolar moments of both sets are endorsed, that is a circular current of any spatial size is a continuum of pairs of distinct Cartesian elemental magnetic dipoles. In addition the customary *ad hoc* definition of magnetic dipole moment is deceptively erroneous, thus prompting a review of many relations involving it. These include the magnetic torque and magnetic field generated by it, and the classical magneto-mechanical ratio.

1. Introduction

This article arose from the humble aim of verifying from first principles, that the new theory of representing any source of magnetic fields as a distribution of elemental magnetic vector charges [1] could reproduce the traditionally acclaimed magnetic properties of or due to a circular electric charge current [2–7]. Further, one is amazed at the traditional definition of the magnetic moment either in terms of a hitherto non-existent scalar quantity (Gilbert model of 1600 or Dirac's magnetic charge of 1931) or in terms of a *complete* current loop only (Ampère model) [2–7]. Both models are not exactly in line with the moments of other physical quantities.

After identifying the various continuous distributions of magnetic vector charges and modeling a circular current as one of these, we demonstrate by deriving from first principles that a circular current has two distinct Cartesian elemental magnetic dipoles which must be fully considered as essential elements in deriving its various magnetic properties, and that the above traditional assumptions are faulty. Here easier evaluation of vector cross-products is due to expressing vectors in terms of Cartesian unit vectors, with the yz-plane as the field plane and the source positions in the xy-plane.

2. Continuously distributed electric currents and magnetic vector charges

Figure 1 shows our models of continuously distributed elements of electric current vectors in elemental spaces of line length $d\ell$, surface area da_{t}, and space volume dv which have respective cross-sections of point $\mathrm{P}_\times$, length $\ell_\times$, area $\boldsymbol{a}_\times$, and spatial current density vectors $\mathbf{I}$, $\mathbf{K}$, $\mathbf{J}$. By harmonized definition the line, surface and volume elemental magnetic vector charges are

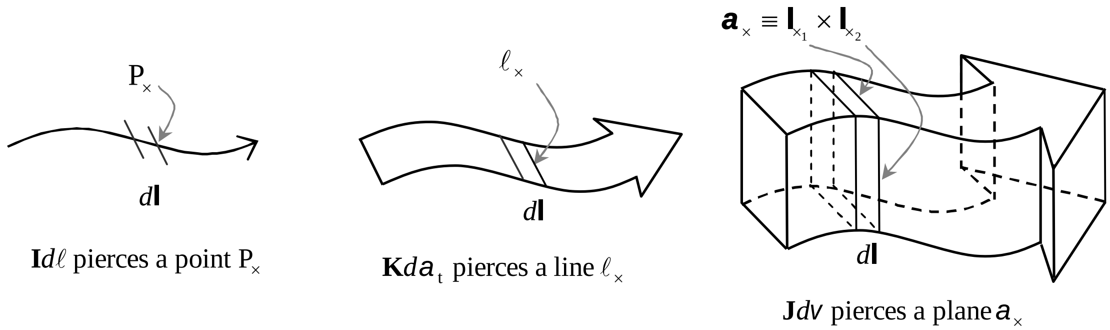

Figure 1. Elemental spaces $d\ell$, da_{t}, dv, cross-sections $\mathrm{P}_\times$, $\ell_\times$, $\boldsymbol{a}_\times$ and *current densities* $\mathbf{I}$, $\mathbf{K}$, $\mathbf{J}$.

$$d\mathbf{Q} = \begin{cases} \mu_0 \mathbf{I} d\ell \equiv \mu_0 I d\mathbf{l}, & \text{along a line} \\ \mu_0 \mathbf{K} da_{\mathrm{t}} = \mu_0 \mathbf{K} \left| \mathbf{l}_\times \times d\mathbf{l} \right| \equiv \left| \mu_0 \mathbf{K} \times \mathbf{l}_\times \right| d\mathbf{l} = \mu_0 I_{\mathbf{l}_\times} d\mathbf{l}, & \text{over a surface} \\ \mu_0 \mathbf{J} dv = \mu_0 \mathbf{J} \left(\boldsymbol{a}_\times \cdot d\mathbf{l} \right) \equiv \left(\mu_0 \mathbf{J} \cdot \boldsymbol{a}_\times \right) d\mathbf{l} = \mu_0 I_{\boldsymbol{a}_\times} d\mathbf{l}, & \text{in a volume} \end{cases} \quad (1)$$

Note the alternative representations on the extreme right hand sides where $\mu_0 I$, $\mu_0 I_{\mathbf{l}_\times}$ and $\mu_0 I_{\boldsymbol{a}_\times}$ are magnitudes of the magnetic vector charges *per unit length* of the line, surface and volume spatial distributions respectively, and in each case the direction is given by the elemental displacement $d\mathbf{l}$.

Traditionally [2–7], I the magnitude of $\mathbf{I}$ is inaptly called current, while *simultaneously* accepting $I\mathrm{d}\mathbf{l}$ as a *current element* (that is, an elemental line current). Then $\mathbf{K}$ and $\mathbf{J}$ are confusingly viewed as "surface" and "volume" densities of I over perpendicular components of the cross-sections $\ell_\times$ and $\boldsymbol{a}_\times$. This is an apparent mix-up between the space occupied and the cross section of that space, due to not recognizing the two alternative representations of current distributions. Rather I should be referred to as the current-flux, so that $\mathbf{I}$, $\mathbf{K}$ and $\mathbf{J}$ can also be called *point, line* and *surface* current-flux densities.

3. Magnetic moments and magnetic torques of the distinct Cartesian magnetic dipoles

Figure 2 shows at points P_1, P_2, P_3 and P_4, on a circle of radius ρ, Cartesian elemental magnetic vector charges [1] perpendicular and parallel to the yz-plane:

$$\pm \hat{\mathbf{x}} dQ_a = \pm \hat{\mathbf{x}} Q_0 \sin\phi \, d\phi \qquad \text{and} \qquad \pm \hat{\mathbf{y}} dQ_b = \pm \hat{\mathbf{y}} Q_0 \cos\phi \, d\phi \quad (2)$$

where $Q_0 = \mu_0 \rho I$. When paired as Cartesian magnetic dipoles, their respective moments [8] are

$$d\boldsymbol{m}_a = d\boldsymbol{m}_{a_+} + d\boldsymbol{m}_{a_-} = \left(\hat{\boldsymbol{\rho}}_1 - \hat{\boldsymbol{\rho}}_2 \right) \rho \times \hat{\mathbf{x}} dQ_a = \mathbf{s}_a \times \hat{\mathbf{x}} dQ_a = -\hat{\mathbf{y}} s_a \times \hat{\mathbf{x}} dQ_a = \hat{\mathbf{z}} dm_a \quad (3a)$$

$$d\boldsymbol{m}_b = d\boldsymbol{m}_{b_+} + d\boldsymbol{m}_{b_-} = \left(\hat{\boldsymbol{\rho}}_2 - \hat{\boldsymbol{\rho}}_3 \right) \rho \times \hat{\mathbf{y}} dQ_b = \mathbf{s}_b \times \hat{\mathbf{y}} dQ_b \equiv +\hat{\mathbf{x}} s_b \times \hat{\mathbf{y}} dQ_b = \hat{\mathbf{z}} dm_b \quad (3b)$$

These Cartesian elemental magnetic dipolar moments are typically distinct as

$$s_a = 2\rho \sin\phi \neq 2\rho \cos\phi = s_b \qquad \text{and} \qquad dm_a = s_a dQ_a \neq s_b dQ_b = dm_b \quad (4)$$

Integrating (3a) and (3b) from ϕ to $\phi + \pi$ yields the magnetic dipolar moments

$$\boldsymbol{m}_a \equiv \hat{\mathbf{z}} m_a = \hat{\mathbf{z}} \pi \rho Q_0 \quad (5a)$$

$$\boldsymbol{m}_b \equiv \hat{\mathbf{z}} m_b = \hat{\mathbf{z}} \pi \rho Q_0 \quad (5b)$$

Thus the overall magnetic dipolar moment $\boldsymbol{m} = \boldsymbol{m}_a + \boldsymbol{m}_b = \hat{\mathbf{z}}2\pi\rho Q_0$ is twice the traditional value for a circular current [2–7].

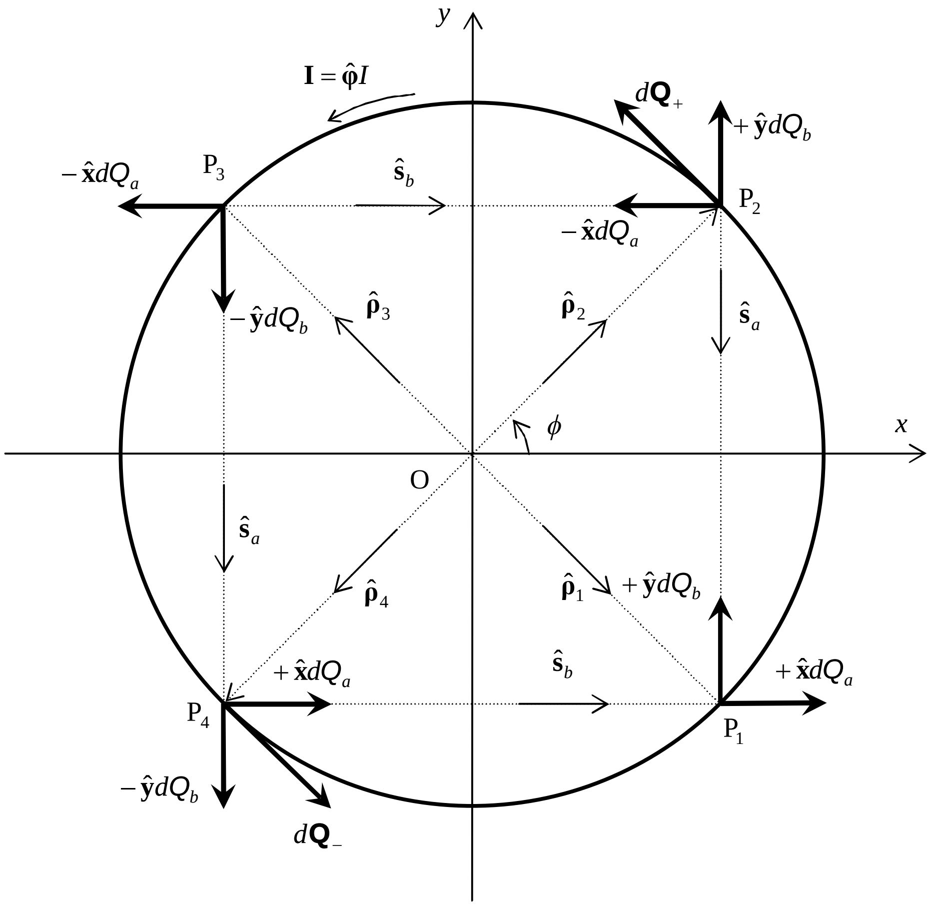

Figure 2. Separated Cartesian elemental magnetic vector charges pair up into magnetic dipoles.

In a magnetic field $\mathbf{H} = \hat{\mathbf{x}}H_x + \hat{\mathbf{y}}H_y + \hat{\mathbf{z}}H_z$, the two Cartesian elemental magnetic dipoles are characterized by magnetic forces $d\mathbf{F}_a = \hat{\mathbf{x}}dQ_a \times \mathbf{H}$ and $d\mathbf{F}_b = \hat{\mathbf{y}}dQ_b \times \mathbf{H}$ and their opposites. Then the coupled elemental magnetic torques [8] on the dipoles are

$$d\boldsymbol{\tau}_a = \mathbf{s}_a \times d\mathbf{F}_a \equiv -\hat{\mathbf{y}}s_a \times (\hat{\mathbf{x}}dQ_a \times \mathbf{H}) = -\hat{\mathbf{y}} \times (\hat{\mathbf{z}}H_y - \hat{\mathbf{y}}H_z)s_a dQ_a = -\hat{\mathbf{x}}H_y dm_a \neq d\boldsymbol{m}_a \times \mathbf{H} \quad (6a)$$

$$d\boldsymbol{\tau}_b = \mathbf{s}_b \times d\mathbf{F}_b \equiv +\hat{\mathbf{x}}s_b \times (\hat{\mathbf{y}}dQ_b \times \mathbf{H}) = +\hat{\mathbf{x}} \times (-\hat{\mathbf{z}}H_x + \hat{\mathbf{x}}H_z)s_b dQ_b = +\hat{\mathbf{y}}H_x dm_b \neq d\boldsymbol{m}_b \times \mathbf{H} \quad (6b)$$

Note that $d\boldsymbol{\tau}_a \neq d\boldsymbol{\tau}_b$ and $\boldsymbol{\tau}_a \neq \boldsymbol{\tau}_b$. Then the overall coupled torque is

$$\boldsymbol{\tau} = \boldsymbol{\tau}_a + \boldsymbol{\tau}_b = -\hat{\mathbf{x}}m_a H_y + \hat{\mathbf{y}}m_b H_x \equiv \boldsymbol{m}_a \times \mathbf{H} \equiv \boldsymbol{m}_b \times \mathbf{H} \neq \boldsymbol{m} \times \mathbf{H} \equiv 2\boldsymbol{\tau} \quad (7)$$

The equivalences are *accidental* and non-basic relations (cannot be used to define torque) because, although $\boldsymbol{m}_a = \boldsymbol{m}_b$, the differentials $d\boldsymbol{m}_a$ and $d\boldsymbol{m}_b$ differ in size and originating elemental dipoles. Thus the traditional choice [2–7] of $\boldsymbol{m}_a$ or $\boldsymbol{m}_b$ as the total magnetic dipolar moment is unjustified.

4. Magnetic vector potentials and fields of the distinct Cartesian magnetic dipoles

In figure 3 the positions of the field point P relative to the source points P_1', P_2' and P_3' are

$$\mathbf{R}_1 = \hat{\mathbf{r}}r - \hat{\boldsymbol{\rho}}_1'\rho', \qquad \mathbf{R}_2 = \hat{\mathbf{r}}r - \hat{\boldsymbol{\rho}}_2'\rho' \qquad \text{and} \qquad \mathbf{R}_3 = \hat{\mathbf{r}}r - \hat{\boldsymbol{\rho}}_3'\rho' \quad (8)$$

These are expressible in terms of r, θ, ρ', ϕ', $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. For $\rho' << r$, all ensuing first order approximations are after binomial expansions of indicated fractions in brackets, and application of equations (3a), (3b) and (4) for the elemental magnetic dipole moments.

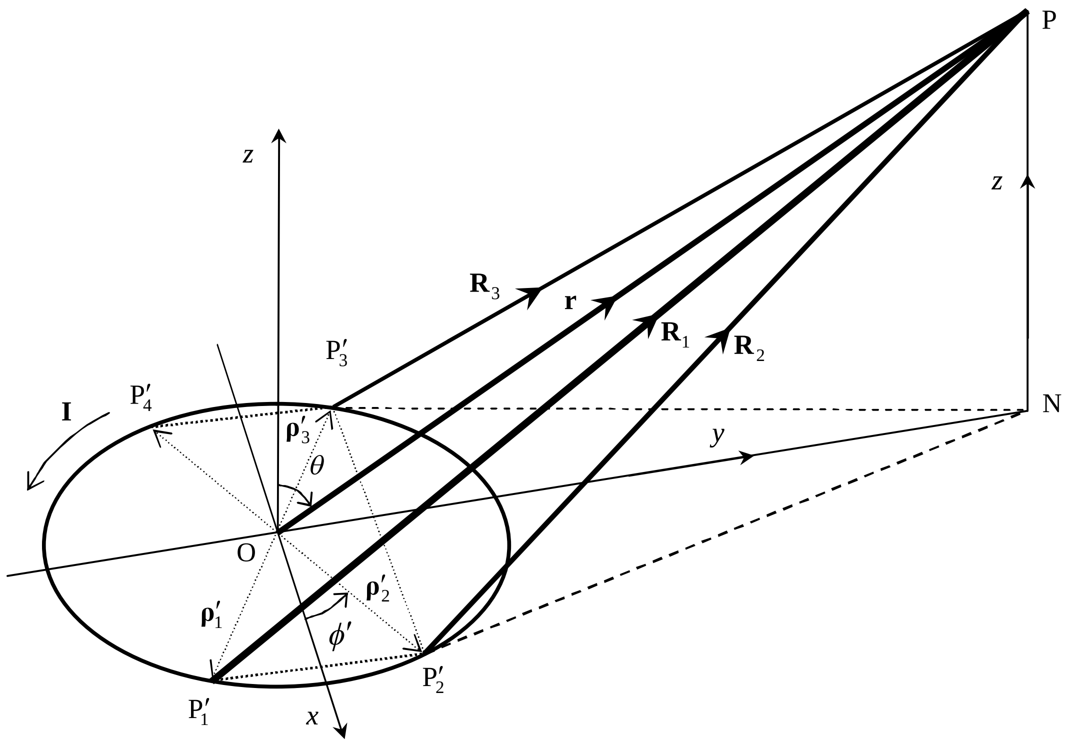

Figure 3. Positions of source points P_1, P_2, P_3 in the xy-plane, and field point P in the yz-plane.

The two distinct Cartesian elemental magnetic dipoles generate at P the harmonized magnetic vector potentials

$$d\mathbf{A}_a = d\mathbf{A}_{a_+} + d\mathbf{A}_{a_-} = \frac{\hat{\mathbf{x}} dQ'_a}{4\pi\mu_0 r}\left(\frac{r}{R_1} - \frac{r}{R_2}\right) \approx \frac{-\hat{\mathbf{x}} dm'_a \sin\theta}{4\pi\mu_0 r^2} \equiv \frac{\hat{\mathbf{z}} dm'_a \times \hat{\mathbf{r}}}{4\pi\mu_0 r^2} \qquad (9a)$$

$$d\mathbf{A}_b = d\mathbf{A}_{b_+} + d\mathbf{A}_{b_-} = \frac{\hat{\mathbf{y}} dQ'_b}{4\pi\mu_0 r}\left(\frac{r}{R_2} - \frac{r}{R_3}\right) = \frac{\hat{\mathbf{y}} dQ'_b}{4\pi\mu_0 r}\left(\frac{r}{R_2} - \frac{r}{R_2}\right) = \mathbf{0} \qquad (9b)$$

This vanishing of $d\mathbf{A}_b$ does not mean an absence of $d\boldsymbol{m}'_b$ [8]. The total magnetic vector potential is

$$\mathbf{A} = \mathbf{A}_a + \mathbf{A}_b = \mathbf{A}_a = \frac{\hat{\mathbf{z}} m'_a \times \hat{\mathbf{r}}}{4\pi\mu_0 r^2} \equiv \frac{\hat{\mathbf{z}} m'_b \times \hat{\mathbf{r}}}{4\pi\mu_0 r^2} = \frac{\hat{\boldsymbol{\varphi}} m'_a \sin\theta}{4\pi\mu_0 r^2} \qquad (10)$$

Traditional wisdom [7] uses (10) to assign $\boldsymbol{m}'_a$ [2–7] the status of "magnetic dipole moment of the loop", thus inadvertently ignoring existence of $\boldsymbol{m}'_b$. Such a conclusion can be likened to stating that, since $\mu_0 \boldsymbol{\mu}_s \cdot \mathbf{H} = \mu_0 \mu_{s_z} H$ is the electron's spin magnetic energy, then the electron's total spin magnetic dipole moment μ_s reduces to its z-component μ_{s_z}!

The magnetic fields at point P due to the two distinct Cartesian elemental magnetic dipoles are

$$d\mathbf{H}_a = d\mathbf{H}_{a_+} + d\mathbf{H}_{a_-} = \frac{\hat{\mathbf{x}} dQ'_a}{4\pi\mu_0 r^3} \times \left(\mathbf{R}_1 \frac{r^3}{R_1^3} - \mathbf{R}_2 \frac{r^3}{R_2^3}\right) \approx \frac{dm'_a}{4\pi\mu_0 r^3}\left\{3\sin\theta\left(\hat{\mathbf{y}}\cos\theta - \hat{\mathbf{z}}\sin\theta\right) + \hat{\mathbf{z}}\right\} \qquad (11a)$$

$$d\mathbf{H}_b = d\mathbf{H}_{b_+} + d\mathbf{H}_{b_-} = \frac{\hat{\mathbf{y}} dQ'_b}{4\pi\mu_0 r^3} \times \frac{\mathbf{R}_2 - \mathbf{R}_3}{(1+\eta_-)^{3/2}} \approx \frac{dm'_b}{4\pi\mu_0 r^3}\hat{\mathbf{z}} \tag{11b}$$

Integrating and then transforming unit vectors from Cartesian to spherical system yields

$$\mathbf{H}_a = \frac{m'_a}{4\pi\mu_0 r^3}\left(\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}} 2\sin\theta\right) \tag{12a}$$

$$\mathbf{H}_b = \frac{m'_b}{4\pi\mu_0 r^3}\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right) \tag{12b}$$

Then since $m'_a = m'_b$ (equations 5a and 5b), the overall magnetic field becomes

$$\mathbf{H} = \mathbf{H}_a + \mathbf{H}_b = \frac{m'_a}{4\pi\mu_0 r^3}\left(2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \equiv \frac{m'_b}{4\pi\mu_0 r^3}\left(2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \tag{13}$$

This cannot justify taking $\boldsymbol{m}'_a$ or $\boldsymbol{m}'_b$ as the circular current's only magnetic moment. Also applying the relation $\mathbf{H} = \nabla \times \mathbf{A} \equiv \nabla \times \mathbf{A}_a$ on (10) to get (13), tradition [7] barely "evades" this scrutiny.

5. Conclusions

Depicting an electric charge current in a circle as a continuum of paired Cartesian elemental magnetic vector charges, parallel and normal to the field plane perpendicularly bisecting the circle, provides realistic models of the structure and attributes of an elemental magnetic dipole. It shows that, contrary to tradition, a circular current, whatever its spatial size, consists of two continuous distributions of distinct Cartesian elemental magnetic dipoles. Each has a distinct nonzero contribution to the overall magnetic dipole moment (which is twice the renowned traditional value), the magnetic field and magnetic torque. Characteristically, the total magnetic vector potential is entirely due one of the continuous distributions of Cartesian elemental magnetic dipoles. Other fundamental outcomes of the above theory include:

(1) Doubling of the classical magneto-mechanical ratio which agrees with Dirac's relativistic electron theory [3].
(2) The traditional analogies between the structures and torques of electric and magnetic dipoles are deceptively erroneous.
(3) Quite generally relations involving the traditional magnetic moment need to be looked at.

References

[1] Chirwa M 2009 A New theory of elemental electromagnetic scalar and vector charges *Brochure, 54th Annual Conference of South African Institute of Physics*, (6-10 July 2009, Durban) Abstract #73.

[2] Jackson J D 1975 *Classical Electrodynamics*, 2nd edition (New York, John Wiley & Sons) pp.178–186, ISBN 0 471 43132 X.

[3] Griffiths D J 1999 *Introduction to Electrodynamics*, 3rd edition (London, Prentice–Hall) pp. 242–258, ISBN 0 13 919960 8.

[4] Pollack G L and Stump D R 2002 *Electromagnetism* (Cape Town, Addison-Wesley) pp. 284–288, ISBN 0 8053 8567 3.

[5] Wangsness R K 1986 *Electromagnetic Fields*, 2nd edition (New York, John Wiley & Sons) pp. 297–308, ISBN 0 471 85912 5.

[6] Panosfky W K H and Phillips M 1962 *Classical Electricity and Magnetism* (London, Addison-Wesley) pp. 128–133, ISBN 0 201 05702 6.

[7] Lorrain P, Corson D P and Lorrain F 1987 *Electromagnetic Fields and Waves*, 3rd edition (New York, W H Freeman & Co) pp. 331–341, ISBN 0 7167 1869 3.

[8] Chirwa M 2010 Status of electric and magnetic dipole moments in a complete characterization of electromagnetic first order moments *Proc. 2nd WSU International Research Conference* (East London, 25th – 27th August 2009) pp. 88–96, ISBN 978 0 620 47102 2.