The Neutron Superfluid in the Interior of Neutron Stars

G.J. Kemp[*], **C.A. Engelbrecht**[*], **F.A.M Frescura**[†]

[*]University of Johannesburg, [†]University of Witwatersrand

E-mail: `garry@kemp.za.org`

Abstract. Feynman, in his work on superfluid helium, attempted both qualitative and quantitative descriptions of superfluid helium's excitation spectrum. We briefly review Feynman's approach to superfluid helium and assess the applicability of this theory to neutron star interiors. We find that Feynman's calculations for superfluid helium may indeed be applied to the neutron superfluid in a neutron star.

1. Introduction

F. London proposed that the onset of superfluidity in liquid helium was a result of Bose-Einstein condensation [1], [2]. Tisza then developed this idea into his so-called two-fluid model [2], [3]. Landau, however, critisised this view-point. He argued that atoms in the condensate would still collide with excited atoms generating friction amongst the atoms in the condensate, thus destroying the superfluidity [4]. Starting from quantum hydrodynamics, Landau proposed his own two-fluid model for liquid helium below the λ-point. Thus, there were two opposing view-points regarding the quantum liquid - that of London's and that of Landau's.

An atomic theory of liquid helium below the λ-point was successfully tackled by Feynman [5], [6], [7] and [8]. His theory describing the transition as well as the excitation spectrum agreed both qualitatively and quantitatively with experiment. For example, his path-integral approach to liquid helium, modeled as a system of non-interacting bosons, predicts a transition in the system roughly where the observed transition occurs.

In section 2, we review Feynman's path-integral treatment of liquid helium. We do not review his arguments for the transition being attributed to the statistics of the system however. In section 3, we review Feynman's approach to predicting the shape of the excitation spectrum as well as the energy gap Δ. We conclude with a short discussion of how this calculation may be reproduced for the neutron superfluid of a neutron star.

2. The Transition

We begin by writing down the partition function for an interacting system of N bosons in the path-integral approach to quantum mechanics. Denote the particle coordinates at time β by $\mathbf{r}_i(\beta)$ where i labels the particle and runs from 1 to N. Suppose that the initial and final configurations of the system at times $\beta = 0$ and β respectively, were identical: $(\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_N)$. Then $\mathbf{r}_i(0) = \mathbf{z}_i$ and $\mathbf{r}_i(\beta) = \mathbf{z}_i$. The interaction forces are two-body forces where there is a weak attraction at the average atomic distance and a strong repulsion when the atoms approach closer

than 2.6A. He^4 atoms obey Bose-Einstein statistics. Thus, we cannot distinguish between the configuration $\mathbf{r}_i(\beta) = \mathbf{z}_i$ and a configuration in which the atoms have been permuted amongst the positions. Denote by P a permutation of atoms amongst the $\mathbf{z}_i$ positions. Then, configurations $\mathbf{r}_i(\beta) = \mathbf{z}_i$ and $\mathbf{r}_i(\beta) = P\mathbf{z}_i$ are indistinguishable. Thus, the exact partition function for the system is

$$Q = \frac{1}{N!} \sum_P \int d^N \mathbf{z} \int_{tr_P} \mathcal{D}^N[\mathbf{r}(u)] \exp \left\{ -\int_0^\beta \left[\frac{m}{2\hbar^2} \sum_i \left(\frac{d\mathbf{r}_i}{du} \right)^2 + \sum_{ij} V\mathbf{r}_i - \mathbf{r}_j) \right] du \right\}$$

where $\int_{tr_P}$ means that we integrate over all particle trajectories for a given permutation P and we sum over all possible permutations P.

The atoms are then modeled as hard impenetrable spheres preventing the atoms from overlapping. We approximate the atoms as free particles each with an effective mass m'. The partition function in this model may be approximated by

$$Q = \frac{K_\beta}{N!} \sum_P \int \left(\frac{m'}{2\pi\beta\hbar^2} \right)^{3N/2} \exp \left[-\frac{m'}{2\beta\hbar^2} \sum_i (\mathbf{z}_i - P\mathbf{z}_i)^2 \right] \rho(\mathbf{z}_1, ..., \mathbf{z}_N) \, d^N \mathbf{z} \qquad (1)$$

where $\rho(\mathbf{z}_1, ..., \mathbf{z}_N)$ is a weighting factor for the configuration $(\mathbf{z}_1, ..., \mathbf{z}_N)$; ρ should be nearly zero if atoms overlap and larger if the atoms are well spaced from each other. Thus, for low temperatures (large β), ρ is interpreted as the probability density of the ground state wavefunction. Approximately

$$\rho(\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_N) = \left\{ \begin{array}{ll} 0 & |\mathbf{z}_i - \mathbf{z}_j| > b \\ 1 & |\mathbf{z}_i - \mathbf{z}_j| < b, \end{array} \right.$$

where b is the diameter of impenetrable atoms of a classical gas.

It may be shown that the partition function in (1) approaches the correct limit for large and small values of β [5].

Feynman then shows that (1) describing a non-interacting system of helium atoms of mass m' obeying Bose-Einstein statistics is capable of predicting the experimentally observed transition. Thus, Feynman, from first principles, showed that London's view-point is essentially correct; the observed transition at 2.19K is indeed a result of the Bose statistics of the system. From the approximations made, however, the partition function (1) is not able to predict the order of the transition nor the temperature dependence of the specific heat of HeI as the transition is approached.

3. The Excitation Spectrum

3.1. The Lowest Excitations

The lowest excited states of liquid helium involve large groups of atoms or long wavelengths, i.e., compression waves [6]. Suppose that there is a small spike in the density of the fluid in some large region along with a rarefaction adjacent to this region. For equilibrium to establish itself, atoms must move from the high density region to the low density region until the densities are equal. Establishing this equilibrium involves the motion of a relatively large mass of fluid which can then have low kinetic energy. If c is the speed of sound in the liquid, k a wavenumber of these waves and $\omega = ck$ the frequency, then the excitation energy $\hbar ck$ can be arbitrarily small. The first excited states above the ground state are therefore phonons. This conclusion is not exclusively applicable to liquid helium and may apply equally well to the lowest excited states of a neutron superfluid.

3.2. The Minimum In The Spectrum

For atoms obeying Bose statistics, motions on an atomic scale require a minimum energy of excitation above the ground state [7]. The nature of these excited states are speculated to be a small rotating ring of atoms or the rapid motion of an atom through the liquid. The latter excitation involves other atoms in front moving out of its way and closing in behind it. The last possibility is the "excitation of an atom in the local cage formed around it by its neighbors" [7]. For atoms obeying Bose statistics, the wavefunction describing each of the suggested excitations is of the form [7]:

$$\psi = \sum_i f(\mathbf{r}_i)\phi \tag{2}$$

where $f(\mathbf{r})$ is some function of the position and the sum is taken over all particles in the fluid. The form of function $f(\mathbf{r})$ is found through the variational principle. The Hamiltonian of the system is taken as

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + V - E_0$$

where V is the potential energy and E_0 is the ground state energy. E_0 is subtracted because we are only interested in energies above the ground state. In the variational principle, we select the function that minimizes the quantity

$$E = \frac{\int \psi^* \hat{H} \psi \, d^N\mathbf{r}}{\int \psi^* \psi \, d^N\mathbf{r}}. \tag{3}$$

The solution is found to be

$$f(\mathbf{r}) = e^{i\boldsymbol{k}\cdot\mathbf{r}},$$

with energy

$$E(k) = \frac{\hbar^2 k^2}{2mS(k)} \tag{4}$$

where $S(k)$, known as the *structure factor*, is the Fourier transform of the liquid's correlation function:

$$S(\mathbf{k}) = \int p(\mathbf{r})e^{i\boldsymbol{k}\cdot\mathbf{r}} \, d^3\mathbf{r}.$$

The quantity $p(\mathbf{r})d^3\mathbf{r}$ is the probability of finding a helium atom within a volume $d^3\mathbf{r}$ containing the point $\mathbf{r}$ if another atom is known to be at $\mathbf{r} = 0$.

The structure factor may be determined from neutron scattering experiments. Starting from $k = 0$, $S(k)$ rises linearly as $\hbar k/2mc$ implying that $E(k)$ rises as $\hbar kc$. Thus, with the wavefunction (2), Feynman is able to reproduce exactly the phonon part of the spectrum. $S(k)$ then reaches a maximum near $k_0 = 2\text{A}^{-1}$ thus producing a minimum in $E(k)$ for this k value. Beyond this value, $S(k)$ falls to approach unity while undergoing small oscillations. Thus, eventually $E(k)$ will increase according to $\hbar^2 k^2/2m$.

The form of the wavefunction (2) was determined purely on the basis of the particles obeying Bose-Einstein statistics. Thus, its form is not specific to liquid helium and would apply equally

well to any other liquid whose atoms obeyed the same statistics. The only quantity thus far that is liquid helium specific is the structure factor $S(k)$. For liquid helium, the presence of the minimum near k_0, predicted by Feynman's calculation, agrees with experiment and with Landau's theory of liquid helium. In this region, we may therefore write $E(k)$ in the form

$$E(k) = \Delta + \frac{\hbar^2}{2\mu}(k - k_0)^2,$$

where μ is a constant determining the curvature the spectrum and Δ is the energy gap - the minimum energy needed for the excitation. The excitations in this region of the spectrum correspond to Landau's rotons. Qualitatively, (4) thus displays the same behaviour as Landau's curve (see [9]) and is exact for the phonon part of the spectrum. However, while Feynman's curve exhibits a minimum where Landau's does (around k_0), his Δ value is almost double that of Landau's. Landau obtains

$$\frac{\Delta}{k_B} = 8.65 \pm 0.04\text{K}$$

(see [10]) while this fraction computed from (4) is 19.1K [8]. It is suggested that the error in Δ is due to the inaccuracy of the trial wave function ψ, even though it was argued to be a relatively good first approximation. One could lower the value of Δ, for instance, by using a new trial wave function describing the physics of the system more accurately.

The new proposed trial wave function is [8]:

$$\psi = \phi \sum_i \exp(i\boldsymbol{k} \cdot \mathbf{r}_i) \exp\left[i \sum_{j \neq i} g(\mathbf{r}_{ji})\right]. \tag{5}$$

This ψ differs from that in (2) by the exponential factor containing the function $g(\mathbf{r})$. It is argued in [8] that the energy in the variational principle (4) would be lowered if the atoms executed some pattern of flow around the excitation as it flowed through the liquid. The pattern discussed is similar to the "backflow" discussed earlier. The additional exponential factor $\exp[i \sum_{j \neq i} g(\mathbf{r}_{ji})]$ is meant to incorporate a "backflow" as the excitation moves through the liquid. The function g may be determined by solving the differential equation that results from the variation of (3) after the substitution of ψ.

Let $E_1(k)$ and $E_2(k)$ represent the energy spectra found by substituting the old and new wave functions, (2) and (5) respectively, into (4). Values of the ratio $E_2(k)/E_1(k)$ for a range of k values are tabulated in Table II in [8]. There also is a column comparing $E_2(k)/k_B$ with k and it is estimated in [8] that $k = 1.85\,\text{Å}^{-1}$ corresponds to a minimum of 12.00K - a dramatic improvement over $19, 1$K. If additional minor approximations are made in minimizing E in (4), the energy $E_2(k)/k_B$ at $k = 1.85\,\text{Å}^{-1}$ is lowered further to 11.5K.
Recall that for small k, the spectrum $E_1(k)$ resulting from (2) was exact. Thus, there is nothing for the new wave function to improve upon in this region. If this new theory is to be correct, $E_2(k)$ must reduce to $E_1(k)$ as k becomes small and reaches the phonon part of the spectrum. It is verified in [8] that this is indeed the case.
The actual calculation of (4) with (5) is long and difficult. In the interest of simplifying the calculation, (5) was approximated by

$$\psi \approx \phi \sum_i \exp(i\boldsymbol{k} \cdot \mathbf{r}_i)\left[1 + \sum_{j \neq i} g(\mathbf{r}_{ji})\right].$$

This makes the integrals resulting from the variational principle tractable. The quantities used in this calculation which depend on the liquid helium system are the structure factor, the correlation functions and the probability density functions from one, $\rho(\mathbf{r}_1)$, up to four particles $\rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$. With the approximations made during the calculations, $\Delta/k_B = 11.5\text{K}$ is considered in [8] to be accurate to within 0.6%.

4. Discussion

The form of the wavefunctions proposed could apply to any system of bosons in Feynman's model, not only liquid helium. Consider a superfluid composed of paired neutrons in the $^1S_0$ channel. Here, the paired neutrons are the bosons. The same initial wavefunction (2) may be proposed to describe an excitation within the fluid. The mass in the spectrum (4) would be modified to $2m_n$ where m_n is the neutron mass. $S(k)$ in (4) would also be modified to that of the neutron superfluid. The correlation functions, contained in the structure factors, and density functions were the only liquid helium specific properties used in these calculations - the main requirement was that the system obeyed Bose-Einstein statistics. It is therefore conceivable that the calculations presented in [7] and [8] may be reproduced once the corresponding properties for the neutron superfluid are known. We plan to investigate this further in future work.

5. References

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